TYPE A NILPOTENT HESSENBERG VARIETIES WITH HESSENBERG FUNCTION $h(i) \leq i+1$

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Abstract. We study the type A nilpotent Hessenberg varieties associated with Hessenberg functions that satisfy $h(i) \leq i+1$. We call these the generalized parabolic Peterson varieties. We show that such varieties can be decomposed into the union of specific generalized parabolic Peterson varieties $\mathrm{Pet}_{\lambda,\alpha}$, such that λ is an integer partition, α is an integer composition, and α is dominated by λ. We prove that when α is dominated by λ, the cardinality of the maximal dimensional components of $\mathrm{Pet}_{\lambda,\alpha}$ equals the Kostka number $\mathcal{K}_{\lambda\alpha}$, and its dimension is determined only by λ and the length of α. We provide a recursive formula for the Poincaré polynomial of $\mathrm{Pet}_{\lambda,\alpha}$. These results partially answer open questions about the geometry of Hessenberg varieties but raise further questions about the representation-theoretic reasons for these facts.

Contents

1. Introduction

Let n be a positive integer. A **complete flag** in $\mathbb{C}^n$ is a nested sequence of subspaces $F_\bullet = (F_0 \subset F_1 \subset \cdots \subset F_n)$ such that $\dim(F_i) = i$ for all $0 \leq i \leq n$. The **flag variety** $\mathrm{Fl}(n)$ is the set of all complete flags in $\mathbb{C}^n$.

Given a complex $n \times n$ matrix X and a Hessenberg function $h : [n] \to [n]$, which is a weakly increasing function such that $h(i) \geq i$ holds for all $i \in [n]$, the **Hessenberg variety** corresponding to X and h is the subvariety of $\mathrm{Fl}(n)$ defined as

$$\mathrm{Hess}(X,h) := \{F_\bullet \in \mathrm{Fl}(n) : XF_i \subset F_{h(i)} \text{ for all } i \in [n]\}.$$

DeMari and Shayman first introduced the Hessenberg variety in 1988 [4]. DeMari, Procesi, and Shayman generalized their definition to all Lie types in 1992 [3]. For a recent survey, see [1]. The dimension and number of components of $\mathrm{Hess}(X,h)$ are known only in certain special cases. Below, we give three examples of Hessenberg varieties: the Springer fiber, the Peterson variety, and the parabolic Hessenberg variety, all of which are relevant to our studies. Let λ be an integer partition of n, α be an integer composition of n, h_α be the Hessenberg function such that $h_\alpha(i) = i$ if i is a partial sum $\alpha_1 + \alpha_2 + \ldots + \alpha_j$ of α, and $h_\alpha(i) = i + 1$ otherwise. Let N_λ be a $n \times n$ nilpotent matrix of Jordan type λ. Also, define $h_\Delta := h_{(n)}$, which is the function that maps each i to $i + 1$ for all $i \in [n - 1]$ and maps n to n.

(1) The **Springer fiber** $\mathcal{B}_\lambda$ is the Hessenberg variety corresponding to N_λ and the Hessenberg function $h_{\mathrm{id}} : i \mapsto i$ for all $i \in [n]$. Spaltenstein showed that $\mathcal{B}_\lambda$ is pure dimensional with dimension $\sum_i (i-1)\lambda_i$, and its components are in bijection with the standard Young tableaux of shape λ, whose cardinality also equals the dimension of the irreducible representation of the symmetric group S_n corresponding to λ [11]. The last result is not coincidental; there is a S_n action on the cohomology of the Springer fiber [12].
(2) The **Peterson variety** is the Hessenberg variety of a regular nilpotent matrix $N := N_{(n)}$, and the Hessenberg function h_Δ. Dale Peterson introduced this variety in a series of lectures on the quantum cohomology of flag varieties in the 1990s. This variety was then studied by Kostant [8] and later by Insko and Yong [7]. Peterson and Konstant's work shows that the Peterson variety is irreducible of dimension $n - 1$.
(3) The **parabolic Hessenberg variety** corresponding to a complex $n \times n$ matrix X and an integer composition α is the Hessenberg variety containing all complete flags $F_\bullet$ that satisfy $XF_i \subset F_i$ if i is a partial sum $\alpha_1 + \alpha_2 + \ldots + \alpha_j$ of α. Precup and Tymoczko studied the parabolic Hessenberg variety and used it to recover, in an elementary way, a result of Steinberg and Shimomura: the Kostka numbers count the maximal dimensional components of Steinberg varieties [10].

In this paper, we relax the condition in the definition of the Peterson variety that N must be regular and define the **generalized Peterson variety** as

$$\mathrm{Pet}_\lambda := \mathrm{Hess}(N_\lambda, h_\Delta) = \{F_\bullet \in \mathrm{Fl}(n) : N_\lambda F_i \subset F_{i+1} \text{ for all } i \in [n-1]\}.$$

When $\lambda = (n)$, $\mathrm{Pet}_{(n)}$ is the Peterson variety. We prove the following result regarding the dimension and components of the generalized Peterson variety Pet_λ.

Proposition 1.1. *The generalized Peterson variety* Pet_λ *has dimension*

$$\dim(\mathrm{Pet}_\lambda) = \sum_i i\lambda_i - \ell(\lambda),$$

and its components of this maximum dimension are in bijection with the semistandard Young tableaux of the shape λ *and content* $\{1, 2, \ldots, \ell(\lambda)\}$.

We prove this fact in Section 5 as a corollary of a much more general result. This result implies that the dimension of $H^{\mathrm{top}}(\mathrm{Pet}_\lambda)$ is equal to that of the irreducible representation of highest weight λ of the special Lie algebra $\mathfrak{sl}_{\ell(\lambda)}$; see [6, Theorem 6.3]. It would be interesting to provide a representation-theoretic reason for this fact. We found the construction most related to this result in [2, Chapter 4], where Chriss and Ginzburg define a subvariety of the partial flag variety using operator N_λ, and show its homology class is a $\mathfrak{sl}_{\ell(\lambda)}$ irreducible module of highest weight λ.

To better understand the components of the generalized Peterson variety Pet_λ, we study the intersection of Pet_λ with the parabolic Hessenberg variety corresponding to N_λ and α. Define the **generalized parabolic Peterson variety** to be

$$\mathrm{Pet}_{\lambda,\alpha} := \mathrm{Hess}(N_\lambda, h_\alpha) = \left\{F_\bullet \in \mathrm{Fl}(n) : N_\lambda F_i \subset F_{h_\alpha(i)} \text{ for all } i \in [n]\right\}.$$

Every Hessenberg variety defined by a nilpotent matrix and a Hessenberg function $h(i) \leq i+1$ is a generalized parabolic Peterson variety. In particular,

(1) When $\alpha = (1,1,\ldots,1)$, $\mathrm{Pet}_{\lambda,(1,1,\ldots,1)}$ is the Springer fiber $\mathcal{B}_\lambda$.
(2) When $\alpha = (n)$, $\mathrm{Pet}_{\lambda,(n)}$ is the generalized Peterson variety Pet_λ.
(3) When $\lambda = (1,1,\ldots,1)$, $\mathrm{Pet}_{(1,1,\ldots,1),\alpha}$ is the flag variety $\mathrm{Fl}(n)$.

Let $\preceq$ be the refinement order on compositions and $\trianglelefteq$ be the dominance order on the partitions. We say that a partition λ dominates a composition α and denote it as $\alpha \trianglelefteq \lambda$ if λ dominates the integer partition obtained by rearranging the entries of α in decreasing order.We say a generalized parabolic Peterson variety $\mathrm{Pet}_{\lambda,\alpha}$ is **admissible** if $\alpha \trianglelefteq \lambda$. We call a composition β a **coarsest** element in a set of integer compositions if β is not a refinement of any other element in the set.

Below are our two main theorems. We let λ be a partition and α be a composition of n.

Theorem 1.2. *When the generalized parabolic Peterson variety* $\mathrm{Pet}_{\lambda,\alpha}$ *is not admissible, let* $(\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}$ *be the set of all the coarsest compositions that are dominated by* λ *and refine* α*. There is a decomposition*

$$\mathrm{Pet}_{\lambda,\alpha} = \bigcup_{\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \mathrm{Pet}_{\lambda,\beta} \tag{1.1}$$

such that no two varieties in the union are contained within one another.

We will call this the **admissible decomposition** of $\mathrm{Pet}_{\lambda,\alpha}$.

Theorem 1.3. *When* $\mathrm{Pet}_{\lambda,\alpha}$ *is admissible, it has* $\mathcal{K}_{\lambda\alpha}$ *different maximum dimensional components of dimension*

$$\dim(\mathrm{Pet}_{\lambda,\alpha}) = \sum_i i\lambda_i - \ell(\alpha), \tag{1.2}$$

where $\mathcal{K}_{\lambda\alpha}$ *denotes the Kostka number and* $\ell(\alpha)$ *is the length of* α.

Our Theorem 5.14 below is a generalized version of Theorem 1.3. Note that by letting $\alpha = (1,1,1,\ldots,1)$, Theorem 1.3 recovers the known result that $\dim(\mathcal{B}_\lambda) = \sum_i (i-1)\lambda_i$, and the number of components of this dimension equals the cardinality of the standard Young tableaux of shape λ [11]. On the other hand,taking $\alpha = (n)$, Theorems 1.2 and 1.3 together imply Proposition 1.1.

To approach Theorem 1.3, we compute the **Poincaré polynomial** of $\mathrm{Pet}_{\lambda,\alpha}$, defined to be

$$\mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha}; t) := \sum_j \dim_{\mathbb{C}} \left(H^j(\mathrm{Pet}_{\lambda,\alpha}; \mathbb{C}) \right) t^j,$$

where $H^j(\mathrm{Pet}_{\lambda,\alpha}; \mathbb{C})$ is the degree j cohomology of $\mathrm{Pet}_{\lambda,\alpha}$. In [14], Tymoczko proved that every type A Hessenberg variety admits an affine paving, which implies that the singular cohomology of $\mathrm{Pet}_{\lambda,\alpha}$ is concentrated in even degrees. Therefore, $\mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha}; q^{1/2})$is a polynomial in $\mathbb{Z}_{\geq 0}[q]$. We find the following recursive formula for $\mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha}; q^{1/2})$:

Theorem 1.4. *When* λ *is a partition and* α *is a composition of* n*, let* $\alpha^{(i)}$ *be the integer composition obtained by replacing the first entry* α_1 *of* α *with* $\alpha_1 - i$*. For every integer partition* $\lambda \neq (0)$*, we have*

$$\mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha}; q^{1/2}) = \sum_{i=1}^{\alpha_1} \sum_{\mu} f_{\lambda/\mu}(q) \cdot \mathrm{Poin}(\mathrm{Pet}_{\mu,\alpha^{(i)}}; q^{1/2}),$$

where μ runs over every integer partition such that $|\mu| = |\lambda| - i$, λ/μ is a horizontal strip, and $f_{\lambda/\mu}(q)$ is a polynomial in $\mathbb{Z}[q]$ defined in Section 5 below.

We prove this theorem in Section 5 using a counting method developed by Escobar, Precup, and Shareshian in [5].

The structure of this paper is as follows. In Section 2, we introduce the notation and definitions that we will use in our proofs. In Section 3, we develop results on the linear algebra of cyclic subspaces. In Section 4, we present the method to decompose the generalized parabolic Peterson variety and prove Theorem 1.2. In Section 5, we provide the recursive formula to calculate the Poincaré polynomial of $\mathrm{Pet}_{\lambda,\alpha}$ and prove Theorem 1.3, Theorem 1.4, and Proposition 1.1. In the Appendix, we list notation and definitions used throughout the paper for the reader's reference.

2. Preliminaries

We introduce the notations used to state and prove our main results. Let n be a positive integer and $[n] = \{1, 2, \dots, n\}$.

2.1. **Integer composition and refinement order.** An **integer composition** of n, or more succinctly, a **composition** of n, is a list of positive integers $\alpha = (\alpha_1, \dots, \alpha_\ell)$ such that $\sum_i \alpha = n$. We use the letters α, β and γ for integer compositions in this paper. The number of positive entries in α is called the **length** of α and is denoted by $\ell(\alpha)$. The **size** of α is the sum of the entries of α, which is denoted by $|\alpha|$. We make the convention that $\alpha_i = 0$ if $i > \ell(\alpha)$ and $\alpha_0 := 0$. We say that (0) is the unique integer composition of 0, and its length is 0.

For every positive integer j, define the j-th **partial sum** $ps_j(\alpha)$ of α to be $ps_j(\alpha) := \sum_{i=1}^{j} \alpha_j$, and the **partial sum set** $PS(\alpha)$ of α to be the set of all partial sums of α. Notice that the cardinality of $PS(\alpha)$ is equal to $\ell(\alpha)$. The map $\alpha \mapsto PS(\alpha)$ is a bijection between the set of integer compositions of n and the subsets of $[n]$ that contain n.

Example 2.1. *Let $\alpha = (2, 3, 1, 2)$. We have $|\alpha| = 8$ and $\ell(\alpha) = 4$. The third partial sum $ps_3(\alpha)$ is $2 + 3 + 1 = 6$ and the partial sum set of α $PS(\alpha) = \{2, 5, 6, 8\}$.*

We impose the **refinement order** on the set of compositions of the same size, as defined below. Given integer compositions α and β such that $|\alpha| = |\beta|$, we say that α is a **refinement** of β (or α **refines** β) and write $\alpha \preceq \beta$ if $PS(\beta) \subseteq PS(\alpha)$. We use $\mathcal{R}_\alpha$ to denote the set of all integer compositions of size $|\alpha|$ that refine α. The composition $(1, 1 \dots, 1)$ refines every composition of the same size, and every composition of n is a refinement of (n).

Example 2.2. *Let $\alpha = (2, 4, 5)$, $\beta = (2, 2, 2, 2, 3)$, and $\gamma = (2, 3, 1, 5)$. Then, $PS(\alpha) = \{2, 6, 11\}$, $PS(\beta) = \{2, 4, 6, 8, 11\}$, and $PS(\gamma) = \{2, 5, 6, 11\}$. We have $\beta, \gamma \in \mathcal{R}_\alpha$, i.e. $\beta \preceq \alpha$ and $\gamma \preceq \alpha$, because $PS(\alpha)$ is a subset of both $PS(\beta)$ and $PS(\gamma)$. None of β or γ refines the other, since $PS(\beta) \not\subset PS(\gamma)$ and $PS(\gamma) \not\subset PS(\beta)$.*

Suppose A is a set of compositions that have the same size. For every composition $\alpha \in A$, we call α

- a **coarsest** element in A, if $\alpha \not\preceq \beta$ for every $\beta \in A$ such that $\beta \neq \alpha$;
- a **minimal length** element in A, if $\ell(\alpha) \leq \ell(\beta)$ for every $\beta \in A$.

Accordingly, we define A^{crs} (resp. $A^{\min}$) as the subset of A consisting of all the coarsest (resp. minimal length) elements of A. By definition, all elements in $A^{\min}$ must have the same length. The set $A^{\min}$ is always a subset of A^{crs}, since for any two distinct compositions α and β in A, the condition that $\ell(\alpha) \leq \ell(\beta)$ implies $\alpha \npreceq \beta$.

Example 2.3. *If* $A = \{(5,2),(4,3),(3,4),(2,4,1),(3,3,1),(3,2,1,1)\}$*, then* $A^{\mathrm{crs}} = \{(5,2),(4,3),(3,4),(2,4,1)\}$ *and* $A^{\min} = \{(5,2),(4,3),(3,4)\}$*. Notice that* $A^{\min} \subsetneq A^{\mathrm{crs}}$*, all elements in* $A^{\min}$ *have length* 2*, while there is one element in* A^{crs} *that has length* 3*.*

We define a binary operator $\circ$ on the set of integer compositions of the same size. Given integer compositions α and β such that $|\alpha| = |\beta|$, we define $\alpha \circ \beta$ to be the integer composition such that $PS(\alpha \circ \beta) = PS(\alpha) \cup PS(\beta)$. It follows from the definition that $\alpha \circ \beta$ is the unique coarsest element among all integer compositions that refine both α and β.

Example 2.4. *Let* $\alpha = (3,3,2)$ *and* $\beta = (2,2,3,1)$*. Since* $PS(\alpha) = \{3,6,8\}$ *and* $PS(\beta) = (2,4,7,8)$*, we have* $PS(\alpha) \cup PS(\beta) = \{2,3,4,6,7,8\}$*. Therefore,* $(3,3,2) \circ (2,2,3,1) = (2,1,1,2,1,1)$ *which is a refinement of both* $(3,3,2)$ *and* $(2,2,3,1)$*.*

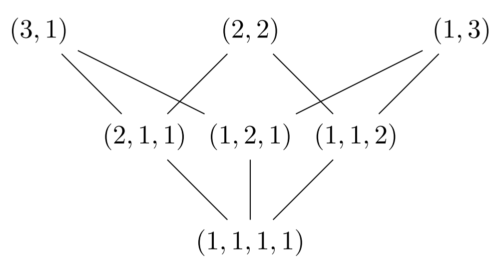
2.2. **Integer partition and dominance order.** An **integer partition** of n, or for short, a **partition** of n, is a list of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that the sum of the entries of λ is n and $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_k$. Integer partitions form a proper subset of the set of integer compositions. We use the letters λ, μ, and ν to denote integer partitions in this paper.

We impose the **dominance order** on the set of all integer partitions of the same size, defined as follows. Given integer partitions λ and μ such that $|\lambda| = |\mu|$, we say that λ **dominates** μ and write $\lambda \trianglerighteq \mu$ if $ps_j(\lambda) \geq ps_j(\mu)$ for all $j \geq 1$. The partition (n) dominates every partition of the same size, and the partition $(1,1,\ldots,1)$ is dominated by every partition of n.

Given a partition λ and a composition α such that $|\alpha| = |\lambda|$, we say that α is **dominated by** λ and write $\lambda \trianglerighteq \alpha$ if $\lambda \trianglerighteq \mu$, where μ is the integer partition obtained by rearranging the entries of α in decreasing order. We use $\mathcal{D}_\lambda$ to denote the set of all integer compositions that are dominated by λ. From the definition, it follows that every composition $\alpha \in \mathcal{D}_\lambda$ satisfies $\ell(\alpha) \geq \ell(\lambda)$.

Example 2.5. *Let* $\lambda = (3,1)$*. The partitions dominated by* $(3,1)$ *are* $(3,1),(2,2),(2,1,1)$*, and* $(1,1,1,1)$*. The coarsest elements among them are* $(3,1)$ *and* $(2,2)$*.*

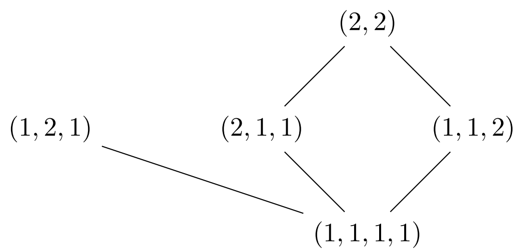
The set $\mathcal{D}_{(3,1)}$ *contains* $(3,1),(2,2),(1,3),(2,1,1),(1,2,1),(1,1,2)$*, and* $(1,1,1,1)$*. Below is the Hasse diagram of elements in* $\mathcal{D}_{(3,1)}$ *under the refinement order.*



In this case, the sets $\mathcal{D}^{\mathrm{crs}}_{(3,1)}$ *and* $\mathcal{D}^{\mathrm{min}}_{(3,1)}$ *are the equal and contain elements* $(3,1)$, $(2,2)$, *and* $(1,3)$.

Example 2.6. *Let* $\lambda = (2,2)$. *The partitions dominated by* $(2,2)$ *are* $(2,2),(2,1,1)$, *and* $(1,1,1,1)$. *The coarsest element among them is* $(2,2)$.

The set $\mathcal{D}_{(2,2)}$ *contains* $(2,2),(2,1,1),(1,2,1),(1,1,2)$, *and* $(1,1,1,1)$. *Below is the Hasse diagram of elements in* $\mathcal{D}_{(2,2)}$ *under the refinement order.*



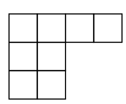
In this case, the set $\mathcal{D}^{\mathrm{crs}}_{(2,2)}$ *contains* $(2,2)$ *and* $(1,2,1)$, *while the set* $\mathcal{D}^{\mathrm{min}}_{(2,2)}$ *only contains* $(2,2)$.

2.3. **Horizontal strips.** The **Young diagram** is a finite collection of boxes arranged in left-justified rows, with the row lengths in non-increasing order. The **shape** of a Young diagram is a partition λ such that λ_i is the number of boxes in the i-th row of the Young diagram. We denote the **transpose partition** of λ as λ^*, which is defined to be the partition such that

$$\lambda^*_j = \#\,\{\text{ Boxes in the } j\text{-th column of the Young diagram of } \lambda\}.$$

By definition, λ^*_j also equals the number of entries of λ that are greater than or equal to j.

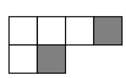
Example 2.7. *If* $\lambda = (4,2,2)$, *then* λ *has length* 3 *and the Young diagram of* λ *is the following.*



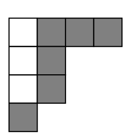
Its transpose partition λ^* *is* $(3,3,1,1)$, *since there are* 3 *boxes in the first and second columns of* λ *and* 1 *box in the third and fourth columns of* λ.

Let λ and μ be different partitions such that $\lambda_i \geq \mu_i$ for all i. The **skew Young diagram** of shape λ/μ is the collection of boxes in the Young diagram of shape λ that do not appear in the Young diagram of shape μ. We denote the skew Young diagram of shape λ/μ simply by λ/μ.

Example 2.8. *The skew Young diagram for shape* $(4,2)/(3,1)$ *is shown in gray below.*



Example 2.9. *The skew Young diagram for the shape* $(4,2,2,1)/(1,1,1)$ *is shown in gray below.*



Our arguments use a specific class of skew Young diagrams called horizontal strips. A skew Young diagram of shape λ/μ is called a **horizontal strip** if it contains at most one box in each column. In other words, λ/μ is a horizontal strip if and only if $0 \leq \lambda_i^* - \mu_i^* \leq 1$ for all i. Equivalently, λ/μ is a horizontal strip if and only if we have $\lambda \neq \mu$ and $\lambda_i \leq \mu_i \leq \lambda_{i+1}$ for all $i \in [\ell(\lambda)]$. When λ/μ is a horizontal strip, we let $I_{\lambda/\mu}$ be the set that contains all the column indices of the boxes in the skew Young diagram λ/μ. In other words, $I_{\lambda/\mu}$ is the set of all indices j such that $\lambda_j^* - \mu_j^* = 1$. In Examples 2.8 and 2.9, the skew tableau $(4,2)/(3,1)$ is a horizontal strip, and $I_{(4,2)/(3,1)} = \{2,4\}$. The skew tableau $(4,2,2,1)/(1,1,1)$ from Example 2.9 is not a horizontal strip. We make the convention that when λ is of length 1, then $\lambda/(0)$ is a horizontal strip that has the same shape as the Young diagram of λ. For example, $(4)/(0)$ is the horizontal strip shown in gray below.

For every partition λ, we let $\Gamma(\lambda)$ be the set of all partitions μ such that λ/μ is a horizontal strip. If m is an integer, we let $\Gamma(\lambda, m)$ be the set of all partitions μ such that λ/μ is a horizontal strip and $|\mu| = m$

Example 2.10. *When $\lambda = (3,1)$, we have*

$$\begin{aligned}
\Gamma\big((3,1)\big) &= \big\{(3,1),(3),(2,1),(2)\big\},\\
\Gamma\big((3,1),2\big) &= \big\{(2),(1,1)\big\},\\
\Gamma\big((3,1),3\big) &= \big\{(3),(2,1)\big\}.
\end{aligned}$$

2.4. **Semistandard Young tableau.** A **semistandard Young tableau** of shape λ and content α is a filling of the Young diagram of shape λ with numbers $1, 2, \ldots, \ell(\alpha)$ such that i appears α_i times, and the filling weakly increases along each row and strictly increases down each column.

We denote the set of all semistandard Young tableaux of shape λ and content α by $\mathrm{SSYT}(\lambda, \alpha)$. The **Kostka number** $K_{\lambda\alpha}$ is the cardinality of $\mathrm{SSYT}(\lambda,\alpha)$. It is well-known that $K_{\lambda\alpha} > 0$ if and only if $\lambda \trianglerighteq \alpha$; see [6, Subsection 4.3]. We take the convention that $\mathcal{K}_{(0)(0)} = 1$.

When ℓ is an integer, let $\mathrm{SSYT}(\lambda, \ell)$ denote the set of all semistandard Young tableaux whose content α satisfies $\ell(\alpha) = \ell$. Since $\mathrm{SSYT}(\lambda,\alpha)$ is not empty if and only if $\alpha \in \mathcal{D}_\lambda$, we have

$$\mathrm{SSYT}(\lambda, \ell) = \coprod_{\substack{\alpha \in \mathcal{D}_\lambda:\\ \ell(\alpha) = \ell(\lambda)}} \mathrm{SSYT}(\lambda, \alpha), \tag{2.1}$$

which implies the equation

$$\#\,\mathrm{SSYT}(\lambda, \ell) = \sum_{\substack{\alpha \in \mathcal{D}_\lambda:\\ \ell(\alpha) = \ell(\lambda)}} \mathcal{K}_{\lambda\alpha}. \tag{2.2}$$

The following two special cases are important to our study.

(1) When $\ell = |\lambda|$, $\mathrm{SSYT}(\lambda, |\lambda|) = \mathrm{SSYT}(\lambda, (1,1,\ldots,1))$ is the set of standard Young tableaux of shape λ. Its cardinality is given by the hook length formula, which equals the dimension of the irreducible representation corresponding to λ of the symmetric group S_n, see [6, Formula 4.12]. For

example, the set SSYT$((4,1),5)$ contains the following 4 elements.

$$\begin{ytableau} 1 & 2 & 3 & 4 \\ 5 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 & 5 \\ 4 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 4 & 5 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 4 & 5 \\ 2 \end{ytableau}$$

Since SSYT$((4,1),5) =$ SSYT$((4,1),(1,1,1,1,1))$, we have

$$\mathcal{K}_{(4,1)(1,1,1,1,1)} = 4.$$

(2) When $\ell = \ell(\lambda)$, the cardinality of SSYT$(\lambda, \ell(\lambda))$ equals

$$\prod_{1 \le i < j \le \ell(\lambda)} \left(\frac{\lambda_i - \lambda_j + j - i}{j - i} \right)$$

which is the dimension of the irreducible representation of highest weight λ of the special Lie algebra $\mathfrak{sl}_{\ell(\lambda)}$, see [6, Theorem 6.3]. For example, the set SSYT$((3,2,1),3)$ contains the following 8 elements.

$$\begin{ytableau} 1 & 1 & 1 \\ 2 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 1 \\ 2 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 3 \\ 2 & 2 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 2 \\ 2 & 2 \\ 3 \end{ytableau}$$

$$\begin{ytableau} 1 & 1 & 3 \\ 2 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 1 & 2 \\ 2 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 2 \\ 2 & 3 \\ 3 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 3 \\ 2 & 3 \\ 3 \end{ytableau}$$

Notice that these elements come from the sets SSYT$((3,2,1),\alpha)$, $\alpha = (3,2,1)$, $(3,1,2)$,$(2,3,1)$, $(2,1,3)$, $(1,3,2)$,$(1,2,3)$, and $(2,2,2)$, which are the compositions of length 3 dominated by $(3,2,1)$. We have
- $\mathcal{K}_{(3,2,1)(3,2,1)}$, $\mathcal{K}_{(3,2,1)(3,1,2)}$, $\mathcal{K}_{(3,2,1)(2,1,3)}$, $\mathcal{K}_{(3,2,1)(2,3,1)}$, $\mathcal{K}_{(3,2,1)(1,3,2)}$, and $\mathcal{K}_{(3,2,1)(1,2,3)}$ are all equal to 1;
- $\mathcal{K}_{(3,2,1)(2,2,2)}$ is equal to 2.

For every composition $\alpha \neq (0)$, let $\overline{\alpha}$ be the composition obtained by deleting the first entry from α. For example, $\overline{(3,1,2)} = (1,2)$ and $\overline{(4)} = (0)$. Clearly, we always have $\ell(\overline{\alpha}) = \ell(\alpha) - 1$. By Pieri's formula; see [6, Formula A.7], we have the following lemma on the recursion of Kostka numbers. A statement equivalent to this lemma, using the language of symmetric functions, can be found in [9, Chapter I, (5.16)].

Lemma 2.11. *For every partition λ and composition α such that $|\lambda| = |\alpha| > 0$, we have that*

$$\mathcal{K}_{\lambda\alpha} = \sum_{\mu \in \Gamma(\lambda, |\overline{\alpha}|)} \mathcal{K}_{\mu\overline{\alpha}} \tag{2.3}$$

Notice that, since the Kostka number $\mathcal{K}_{\lambda,\overline{\alpha}} \neq 0$, if and only if $\overline{\alpha} \trianglelefteq \lambda$, Equation 2.3 is equivalent to the following

$$\mathcal{K}_{\lambda\alpha} = \sum_{\substack{\mu \in \Gamma(\lambda, |\overline{\alpha}|), \\ \overline{\alpha} \trianglelefteq \lambda}} \mathcal{K}_{\mu\overline{\alpha}} \tag{2.4}$$

Example 2.12. *Let $\lambda = (4,2)$ and $\alpha = (2,1,3)$. Then $\overline{(2,1,3)} = (1,3)$, $|\overline{(2,1,3)}| = 4$, and*

$$\Gamma\big((4,2),4\big) = \big\{(4),(3,1),(2,2)\big\}.$$

Then, by Equation (2.3), we have

$$\mathcal{K}_{(4,2)(2,1,3)} = \mathcal{K}_{(4)(1,3)} + \mathcal{K}_{(3,1)(1,3)} + \mathcal{K}_{(2,2)(1,3)}.$$

We can verify that this equation holds by noting that the left-hand side equals

$$\mathcal{K}_{(4,2)(2,1,3)} = \#\left\{ \begin{ytableau} 1 & 1 & 2 & 3 \\ 3 & 3 \end{ytableau},\ \begin{ytableau} 1 & 1 & 3 & 3 \\ 2 & 3 \end{ytableau} \right\} = 2,$$

while the right-hand side also equals

$$\mathcal{K}_{(4)(1,3)} + \mathcal{K}_{(3,1)(1,3)} + \mathcal{K}_{(2,2)(1,3)} = \#\left\{ \begin{ytableau} 1 & 1 & 1 & 2 \end{ytableau} \right\} + \#\left\{ \begin{ytableau} 1 & 2 & 2 \\ 2 \end{ytableau} \right\} + \#\emptyset = 2.$$

Remark 2.13. *We choose $\overline{\alpha}$ to be the map that deletes the first entry from α because it is convenient for our proofs. However, the Kostka number $\mathcal{K}_{\lambda\alpha}$ is invariant under permutations of the entries of α. Thus, if we choose $\overline{\alpha}$ to be the map that deletes any nonzero entry from α, Equation (2.3) still holds.*

The following statement is an immediate corollary of Lemma 2.11.

Corollary 2.14. *For every partition λ and composition α such that $|\lambda| = |\alpha| > 0$, if $K_{\lambda\alpha} > 0$, then there exists some partition $\mu \in \Gamma(\lambda, \overline{\alpha})$ such that $\mathcal{K}_{\mu\overline{\alpha}} > 0$.*

3. Linear algebra of cyclic subspaces

In this Section, we establish some results from linear algebra that we will use in the proofs of the main theorems. We let $\mathbb{K}$ be a field that can be either $\mathbb{C}$ or a finite field $\mathbb{F}_q$. For every partition λ, let N_λ be the nilpotent matrix in Jordan form of Jordan type λ. Let $(e_{ij})_\lambda$, where $i \in [k], j \in [\lambda_i]$, be an orthogonal Jordan basis of N_λ such that

$$N_\lambda e_{ij} = \begin{cases} e_{i(j-1)}, & j > 1 \\ 0, & j = 1. \end{cases}$$

3.1. **Height of a vector.** Throughout this section, let v be a vector in $\mathbb{K}^n$. We also assume that λ is a partition of n.

Definition 3.1. *Given a vector $v \in \mathbb{K}^n$ and a partition λ, if $v \neq 0$, we define the* ***height*** *of v respect to λ as*

$$\mathrm{ht}_\lambda(v) := \max\left\{ d : v \in \mathrm{im}\left({N_\lambda}^{d-1} \right) \right\}.$$

We also make the convention that $\mathrm{ht}_\lambda(0) = \infty$.

We list several properties regarding heights that we will use frequently in the proofs.

Proposition 3.2.
(1) *Let d be a positive integer, for every vector $v \in \mathbb{K}^n$, v is in $\mathrm{im}(N_\lambda^{d-1})$ if and only if $\mathrm{ht}_\lambda(v) \geq d$.*
(2) ***Scale invariance:*** *For every vector $v \in \mathbb{K}^n$ and $c \in \mathbb{K}$ such that $c \neq 0$, we have*
$$\mathrm{ht}_\lambda(cv) = \mathrm{ht}_\lambda(v).$$
(3) ***Shift property:*** *If $v \in \mathbb{K}^n/\{0\}$ and d is a positive integer, we have the inequality:*
$$\mathrm{ht}_\lambda({N_\lambda}^d v) \geq \mathrm{ht}_\lambda(v) + d.$$
(4) ***Valuative property:*** *Given $v_1, v_2, \dots v_\ell \in \mathbb{K}^n$, we have*
$$\mathrm{ht}_\lambda(v_1 + v_2 + \dots v_\ell) \geq \min\{\mathrm{ht}_\lambda(v_1), \dots, \mathrm{ht}_\lambda(v_\ell)\}.$$

Proof. Statements (1) and (2) are obvious from the definition.

(3) Let $\mathrm{ht}_\lambda(v) = \ell$. Notice $v \in \mathrm{im}(N_\lambda^{\ell-1})$. Thus, there must exist some vector w satisfying $X^{\ell-1}w = v$. Furthermore,
$$N_\lambda^d \cdot (N_\lambda^{\ell-1}w) = N_\lambda^{d+\ell-1}w = N_\lambda^d v$$
which implies ${N_\lambda}^d v \in \mathrm{im}({N_\lambda}^{d+\ell-1})$. Therefore, $\mathrm{ht}_\lambda({N_\lambda}^d v) \geq d + \ell = \mathrm{ht}_\lambda(v) + d$.

(4) Let $d := \min\{\mathrm{ht}_\lambda(v_1), \ldots, \mathrm{ht}_\lambda(v_\ell)\}$, we have $v_1, v_2, \ldots, v_\ell \in \mathrm{im}({N_\lambda}^{d-1})$. So, $v_1 + v_2 + \ldots v_\ell \in \mathrm{im}({N_\lambda}^{d-1})$ and $\mathrm{ht}_\lambda(v_1 + v_2 + \ldots v_\ell) \geq d$.

□

Proposition 3.3. *If $v \in \mathrm{im}(N_\lambda)$, there exists a unique vector $\tilde{v} \in \mathbb{K}^n$ such that $N_\lambda \tilde{v} = v$ and $\tilde{v}$ is orthogonal to* $\ker(N_\lambda)$.

Proof. We can write v as $v = \sum_{i,j} c_{ij} e_{ij}$, $i \in [k]$, and $j \in [\lambda_i]$. Since $v \in \mathrm{im}(N_\lambda)$, we have $c_{ij} = 0$ when $j = \lambda_i$. Let $\tilde{v} = \sum_{i,j} c_{ij} e_{i(j+1)}$. Then, $N_\lambda \tilde{v} = v$, and $\tilde{v}$ is orthogonal to $\ker(N_\lambda)$.

To show uniqueness, assume there exists another vector $w \in \mathbb{K}^n$ such that $N_\lambda w = v$ and w are orthogonal to $\ker(N_\lambda)$. Then, $N_\lambda(\tilde{v} - w) = v - v = 0$, which implies $\tilde{v} - w \in \ker(N_\lambda)$. However, since both $\tilde{v}$ and w are orthogonal to $\ker(N_\lambda)$, $\tilde{v} - w$ must be orthogonal to $\ker(N_\lambda)$ as well. Hence, $\tilde{v} - w$ must be 0 and $w = \tilde{v}$. □

Example 3.4. *Let $\lambda = (4, 3, 2)$ and $v = e_{11} + 2e_{12} + 3e_{21}$, then we have $\tilde{v} = e_{12} + 2e_{13} + 3e_{22}$. Also, notice that* $\mathrm{ht}_\lambda(v) = 3$ *and* $\mathrm{ht}_\lambda(\tilde{v}) = 2$.

Proposition 3.5. *Given $v \in \mathrm{im}(N_\lambda)$, let $\tilde{v}$ be the unique vector in $\mathbb{K}^n$ such that $N_\lambda \tilde{v} = v$ and $\tilde{v}$ is orthogonal to* $\ker(N_\lambda)$. *We have the following.*

(1) $\mathrm{ht}_\lambda(\tilde{v}) = \mathrm{ht}_\lambda(v) - 1$.
(2) *For every vector $w \in \mathbb{K}^n$, w satisfies that $N_\lambda w = v$ if and only if w has the orthogonal decomposition $w = \tilde{v} + u$ where $u \in \ker(N_\lambda)$.*
(3) *For every vector $w \in \mathbb{K}^n$ that has the orthogonal decomposition $w = \tilde{v} + u$ where $u \in \ker(N_\lambda)$, we have* $\mathrm{ht}_\lambda(w) = \min\{\mathrm{ht}_\lambda(v) - 1, \mathrm{ht}_\lambda(u)\}$.

Proof. The existence of vector $\tilde{v}$ follows from Proposition 3.3. To show (1), by the shift property, we have $\mathrm{ht}_\lambda(v) = \mathrm{ht}_\lambda(N_\lambda \tilde{v}) \geq \mathrm{ht}_\lambda(\tilde{v}) + 1$. On the other hand, let $d = \mathrm{ht}_\lambda(v)$, and note that the vector $y := \sum_{i,j} c_{ij} e_{i(j+d-1)}$ satisfies ${N_\lambda}^{d-1} y = v$ and also ${N_\lambda}^{d-2} y = \tilde{v}$. So, we also have $\mathrm{ht}_\lambda(\tilde{v}) \geq d - 1$. Therefore, $\mathrm{ht}_\lambda(\tilde{v}) = \mathrm{ht}_\lambda(v) - 1$.

To show (2), if $w = \tilde{v} + u$ for some $u \in \ker(N_\lambda)$, then $N_\lambda w = N_\lambda \tilde{v} + N_\lambda u = v$. On the other hand, if $N_\lambda w = v$, then we have $N_\lambda w = N_\lambda \tilde{v}$, which implies $w - \tilde{v} \in \ker(N_\lambda)$. So, there must exist some $u \in \ker(N_\lambda)$ such that $w = \tilde{v} + u$.

To show (3), For every $w = \tilde{v} + u$, by (1), we have $\mathrm{ht}_\lambda(\tilde{v}) = \mathrm{ht}_\lambda(v) - 1$. So, it suffices to prove that $\mathrm{ht}_\lambda(w) = \min\{\mathrm{ht}_\lambda(\tilde{v}), \mathrm{ht}_\lambda(u)\}$. Notice $\mathrm{ht}_\lambda(v) \geq \min\{\mathrm{ht}_\lambda(\tilde{v}), \mathrm{ht}_\lambda(u)\}$ follows from the valuative property of height. So, it suffices to show $\mathrm{ht}_\lambda(w) \leq \mathrm{ht}_\lambda(\tilde{v})$ and $\mathrm{ht}_\lambda(w) \leq \mathrm{ht}_\lambda(u)$. Notice, by (1) and the shift property of height, we have
$$\mathrm{ht}_\lambda(\tilde{v}) = \mathrm{ht}_\lambda(v) - 1 = \mathrm{ht}_\lambda(N_\lambda w) - 1 \geq \mathrm{ht}_\lambda(w).$$
And since $u = w - \tilde{v}$, by the valuative property again, we have
$$\mathrm{ht}_\lambda(u) \geq \min\{\mathrm{ht}_\lambda(w), \mathrm{ht}_\lambda(-\tilde{v})\} = \min\{\mathrm{ht}_\lambda(w), \mathrm{ht}_\lambda(\tilde{v})\} = \mathrm{ht}_\lambda(w).$$
Therefore, we have $\mathrm{ht}_\lambda(w) \leq \mathrm{ht}_\lambda(\tilde{v})$ and $\mathrm{ht}_\lambda(w) \leq \mathrm{ht}_\lambda(u)$. Hence $\mathrm{ht}_\lambda(w) = \min\{\mathrm{ht}_\lambda(v) - 1, \mathrm{ht}_\lambda(u)\}$. □

For the rest of this subsection, we let ℓ be a positive integer.

Lemma 3.6. *The vector space* $\operatorname{im}(N_\lambda^{\ell-1}) \cap \ker(N_\lambda)$ *has dimension* λ^*_ℓ. *When* $\mathbb{K} = \mathbb{F}_q$, *this space contains* $q^{\lambda^*_\ell}$ *vectors.*

Proof. For every $i \in [\ell(\lambda)]$, we have $e_{i1} \in \operatorname{im}(N_\lambda^{\lambda_i - 1})$ while $e_{i1} \notin \operatorname{im}(N_\lambda^{\lambda_i})$. Therefore, $e_{i1} \in \operatorname{im}(N_\lambda^{\ell-1})$ if and only if $\lambda_i \leq \ell$. By the definition of transpose partition, we also have $\lambda_i \leq \ell$ if and only if $i \geq \lambda^*_i$.

Therefore, the vector space $\operatorname{im}(N_\lambda^{\ell-1}) \cap \ker(N_\lambda)$ is spanned by all e_{i1} such that $i \leq \lambda^*_\ell$. Hence, we have

$$\dim\left(\operatorname{im}(N_\lambda^{\ell-1}) \cap \ker(N_\lambda)\right) = \lambda^*_\ell.$$

When $\mathbb{K} = \mathbb{F}_q$, every vector space of dimension λ^*_ℓ has $q^{\lambda^*_\ell}$ vectors. So, $\operatorname{im}(N_\lambda^{\ell-1}) \cap \ker(N_\lambda)$ has $q^{\lambda^*_\ell}$ vectors. □

Corollary 3.7. *For every positive integer* ℓ, *the set*

$$\{u \in \ker(N_\lambda) : \operatorname{ht}_\lambda(u) = \ell\}$$

is not empty if and only if ℓ *satisfies* $\lambda^*_\ell > \lambda^*_{\ell+1}$. *When* $\mathbb{K} = \mathbb{F}_q$, *this set contains* $q^{\lambda^*_\ell} - q^{\lambda^*_{\ell+1}}$ *vectors.*

Proof. By the definition of height, $\operatorname{ht}_\lambda(u) = \ell$ if and only if $u \in \operatorname{im}(N_\lambda^{\ell-1}) \cap \ker(N_\lambda)$ while $u \notin \operatorname{im}(N_\lambda^{\ell}) \cap \ker(N_\lambda)$. Since $\operatorname{im}(N_\lambda^{\ell}) \cap \ker(N_\lambda)$ is a subspace of $\operatorname{im}(N_\lambda^{\ell-1}) \cap \ker(N_\lambda)$, we can find such u if and only if

$$\lambda^*_\ell = \dim\left(\operatorname{im}(N_\lambda^{\ell-1}) \cap \ker(N_\lambda)\right) > \dim\left(\operatorname{im}(N_\lambda^{\ell}) \cap \ker(N_\lambda)\right) = \lambda^*_{\ell+1}.$$

When $\mathbb{K} = \mathbb{F}_q$, the number of vectors u such that $u \in \operatorname{im}(N_\lambda^{\ell-1}) \cap \ker(N_\lambda)$ while $u \notin \operatorname{im}(N_\lambda^{\ell}) \cap \ker(N_\lambda)$ is $q^{\lambda^*_\ell} - q^{\lambda^*_{\ell+1}}$. □

Example 3.8. *Suppose* $\lambda = (4,2,2,1)$, *then* $\lambda^* = (4,3,1,1)$. *The set*

$$\{u \in \ker(N_\lambda) : \operatorname{ht}_\lambda(u) = \ell\}$$

is not empty if and only if ℓ *is one of* $1, 2$ *or* 4.

When $\mathbb{K} = \mathbb{F}_q$, *we have that:*

- *If* $\ell = 1$, *this set contains* $q^{\lambda^*_1} - q^{\lambda^*_2} = q^4 - q^3$ *vectors.*
- *If* $\ell = 2$, *this set contains* $q^{\lambda^*_2} - q^{\lambda^*_3} = q^3 - q$ *vectors.*
- *If* $\ell = 4$, *this set contains* $q^{\lambda^*_3} - q^{\lambda^*_4} = q - 1$ *vectors.*

In the next lemma, we let d be a positive integer.

Lemma 3.9. *For every nonzero vector* $v \in \operatorname{im}(N_\lambda)$ *such that* $\operatorname{ht}_\lambda(v) = d$, *the set*

$$\{w \in \mathbb{K}^n : N_\lambda(w) = v \text{ and } \operatorname{ht}_\lambda(w) = \ell\}$$

is not empty if and only if

- $\ell = d - 1$, *or*
- $\ell < d - 1$ *while* $\lambda^*_\ell > \lambda^*_{\ell+1}$.

When $\mathbb{K} = \mathbb{F}_q$, *the number of vectors in this set is*

$$\begin{cases} \lambda^*_\ell, & \text{if } \ell = d-1, \\ \lambda^*_\ell - \lambda^*_{\ell+1}, & \text{if } \ell < d-1. \end{cases}$$

Proof. If there doesn't exist any vector v such that $\mathrm{ht}(v) = d$, this statement holds trivially. Otherwise, for every w such that $N_\lambda(w) = v$, by Proposition 3.5, we can write $w = \tilde{v} + u$ for a unique $u \in \ker(N_\lambda)$ and

$$\mathrm{ht}_\lambda(w) = \min\{d-1, \mathrm{ht}_\lambda(u)\}.$$

Therefore, the set $\{w : N_\lambda(w) = v \text{ and } \mathrm{ht}_\lambda(w) = \ell\}$ is not empty only if $\ell \leq d-1$.

If $\ell = d-1$, $\mathrm{ht}_\lambda(w) = \ell$ if and only if $w = \tilde{v} + u$ and $\mathrm{ht}_\lambda(u) \geq d-1 = \ell$, which is equivalent to $u \in \mathrm{im}(N_\lambda^{\ell-1}) \cap \ker(N_\lambda)$. By Lemma 3.6, such u always exists, and when $\mathbb{K} = \mathbb{F}_q$, there are exactly $q^{\lambda_\ell^*}$ such vector u, which correspond to the w such that $N_\lambda(w) = v$ and $\mathrm{ht}_\lambda(w) = d-1$.

If $\ell < d-1$, then $\mathrm{ht}_\lambda(w) = \ell$ if and only if $w = \tilde{v} + u$ and $\mathrm{ht}_\lambda(u) = \ell$. By Corollary 3.7, the set

$$\{u \in \ker(N_\lambda) : \mathrm{ht}_\lambda(u) = \ell\}$$

is not empty if and only if $\lambda_\ell^* > \lambda_{\ell+1}^*$. Therefore, there exists some vector w such that $N_\lambda(w) = v$ and $\mathrm{ht}_\lambda(w) = \ell$ if and only if $\lambda_\ell^* > \lambda_{\ell+1}^*$. When $\mathbb{K} = \mathbb{F}_q$, there are exactly $q^{\lambda_\ell^*} - q^{\lambda_{\ell+1}^*}$ choices for u, which correspond to the choices for w. □

Example 3.10. *Suppose $\lambda = (5,2)$ and $v = e_{11}$. We have $\lambda^* = (2,2,1,1,1)$ and $\mathrm{ht}_\lambda(e_{11}) = 5$. The set*

$$\big\{w : N_\lambda(w) = e_{11} \textit{ and } \mathrm{ht}_\lambda(w) = \ell\big\}$$

is not empty if and only if ℓ is 2 or 4 since $\mathrm{ht}_\lambda(e_{12}) - 1 = 4$, and $\lambda_1^* = \lambda_2^* < \lambda_3^* = \lambda_4^*$. When $\mathbb{K} = \mathbb{F}_q$, we have that:

- *If $\ell = 4$, this set contains $q^{\lambda_4^*} = q$ vectors.*
- *If $\ell = 2$, this set contains $q^{\lambda_2^*} - q^{\lambda_3^*} = q^2 - q$ vectors.*

Example 3.11. *Suppose $\lambda = (6,4,2,1)$ and $v = e_{12}$. We have $\lambda^* = (4,3,2,2,1,1)$ and $\mathrm{ht}_\lambda(e_{12}) = 5$. The set*

$$\big\{w : N_\lambda(w) = e_{12} \textit{ and } \mathrm{ht}_\lambda(w) = \ell\big\}$$

is not empty if and only if ℓ is $4, 2$ or 1 since $\mathrm{ht}_\lambda(e_{12}) - 1 = 4$, and $\lambda_1^* < \lambda_2^* < \lambda_3^* = \lambda_4^*$.

When $\mathbb{K} = \mathbb{F}_q$, we have that:

- *If $\ell = 4$, this set contains $q^{\lambda_4^*} = q^2$ vectors.*
- *If $\ell = 2$, this set contains $q^{\lambda_2^*} - q^{\lambda_3^*} = q^3 - q^2$ vectors.*
- *If $\ell = 1$, this set contains $q^{\lambda_1^*} - q^{\lambda_2^*} = q^4 - q^3$ vectors.*

3.2. **Cyclic subspaces.**

Definition 3.12. *Let $v \in \mathbb{K}^n$. We call the subspace*

$$\langle v \rangle_\lambda := \mathrm{span}\left\{v, N_\lambda v, {N_\lambda}^2 v, \ldots, {N_\lambda}^i v, \ldots\right\}$$

*the **λ-cyclic subspace** generated by v.*

Proposition 3.13. *The dimension of $\langle v \rangle_\lambda$ equals the smallest integer d such that $v \in \ker({N_\lambda}^d)$.*

Proof. Let d be the dimension of $\langle v \rangle_\lambda$. If $v = 0$, this statement is true since the dimension is equal to 0. Otherwise, by the assumption, we have

$$v_1 := v,\ v_2 := N_\lambda v,\ \ldots,\ v_i := {N_\lambda}^{i-1} v,\ \ldots,\ v_d := {N_\lambda}^{d-1} v,$$

all being nonzero vectors; it suffices to prove that they are linearly independent. Suppose there exists $c_1, c_2, \ldots, c_d \in \mathbb{K}$ such that

$$\sum_{i=1}^{d} c_i v_i = c_1 v_1 + c_2 v_2 + \ldots c_i v_i + \ldots c_d v_d = 0.$$

By multiplying ${N_\lambda}^{d-1}$ to both sides, we have $c_1 {N_\lambda}^{d-1} v_1 = c_1 v_d = 0$. So, we must have $c_1 = 0$. Then, by multiplying ${N_\lambda}^{d-2}$ to both sides, we have $c_2 {N_\lambda}^{d-2} v_2 = c_2 v_d = 0$. So, we must have $c_2 = 0$. Continuing this process, we have $c_1 = c_2 = c_3 = \ldots = c_d = 0$. ☐

Lemma 3.14. *For every vector $w \in \langle v \rangle_\lambda$, we have $\mathrm{ht}_\lambda(w) \geq \mathrm{ht}_\lambda(v)$.*

Proof. If $w = 0$, this statement holds trivially. Otherwise, let d be the dimension of $\langle v \rangle_\lambda$. Since $w \in \langle v \rangle_\lambda$, we can write

$$w = \sum_{i=1}^{d} c_i N_\lambda^{i-1} v = c_1 v + c_2 N_\lambda v + \ldots c_i {N_\lambda}^{i-1} v + \ldots + c_d {N_\lambda}^{d-1} v$$

for some coefficients $c_1, c_2, \ldots, c_n$ that cannot all be zero. By the valuative property and scale invariance, we have

$$\mathrm{ht}_\lambda(w) \geq \min\{\mathrm{ht}_\lambda(c_i N_\lambda^{i-1} v) : c_i \neq 0\} = \min\{\mathrm{ht}_\lambda(N_\lambda^{i-1} v) : c_i \neq 0\}.$$

And by the shift property, we have $\mathrm{ht}_\lambda({N_\lambda}^{i-1} v) \geq \mathrm{ht}_\lambda(v)$ for all $1 \leq i \leq d$. Therefore, we have

$$\mathrm{ht}_\lambda(w) \geq \min\{\mathrm{ht}_\lambda({N_\lambda}^{i-1} v : c_i \neq 0\} \geq \mathrm{ht}_\lambda(v)$$

which is the desired inequality. ☐

3.3. **Jordan Chains.**

Definition 3.15. *Given $u \in \ker(N_\lambda)$, a λ-**Jordan chain** containing u is a set*

$$\{w, \quad N_\lambda w, \quad N_\lambda^2 w, \quad \ldots, \quad N_\lambda^{d-2} w, \quad N_\lambda^{d-1} w = u\}, \tag{3.1}$$

*where w is a vector in $\mathbb{K}^n$ that satisfies $N_\lambda^{d-1} w = u$ for some integer d. We call the number of elements in this set the **length** of the chain.*

Remark 3.16. *By the shift property, the elements in a λ-Jordan chain satisfy*

$$\mathrm{ht}_\lambda(w) < \mathrm{ht}_\lambda(N_\lambda w) < \mathrm{ht}_\lambda(N_\lambda^2 w) < \ldots < \mathrm{ht}_\lambda(N_\lambda^{d-2} w) < \mathrm{ht}_\lambda(N_\lambda^{d-1} w) = \mathrm{ht}_\lambda(u).$$

Definition 3.17. *We call a λ-Jordan chain containing u **maximal** if there is no other λ-Jordan chain containing u that has a larger length.*

By the definition of height, we have the following lemma.

Lemma 3.18. *A λ-Jordan chain containing u is maximal if and only if its length equals $\mathrm{ht}_\lambda(u)$.*

The readers can find the following result in standard textbooks; cf [15].

Proposition 3.19. *Every maximal λ-Jordan chain can be extended to become a Jordan basis of N_λ.*

Example 3.20. *If $\lambda = (4,2)$ and $v = e_{11}$. The following is a λ-Jordan chain containing e_{11}:*

$$\{e_{14} + 2e_{13}, \quad e_{13} + 2e_{12}, \quad e_{12} + 2e_{11}, \quad e_{11}\},$$

which has length 4 which equals $ht_\lambda(e_{11})$, so it is maximal. We can extend it to

$$\{e_{14} + 2e_{13}, \quad e_{13} + 2e_{12}, \quad e_{12} + 2e_{11}, \quad e_{11}, \quad e_{21}, \quad e_{22}\},$$

which is a Jordan basis for N_λ.

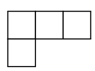
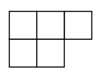
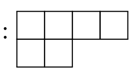
3.4. **The Jordan type of a N_λ on quotient space.**

Definition 3.21. *For every nonzero vector $v \in \mathbb{K}^n$, we let $\lambda(v)$ be the partition of $n - \dim\langle v\rangle_\lambda$ determined by the Jordan type of N_λ restricted to $\mathbb{K}^n/\langle v\rangle_\lambda$.*

Example 3.22. *If $\lambda = (3,2)$ and $v = e_{12}$, then $\langle v\rangle_\lambda = \operatorname{span}\{e_{12}, e_{11}\}$. The nilpotent operator N_λ restricted to $\mathbb{K}^5/\operatorname{span}\{e_{12}, e_{11}\}$ has the Jordan basis $\{e_{13}, e_{21}, e_{22}\}$. Therefore, we have $\lambda(v) = (2,1)$.*

Example 3.23. *If $\lambda = (4,2)$ and $v = e_{12} + e_{21}$, then $\langle v\rangle_\lambda = \operatorname{span}\{e_{12} + e_{21}, e_{11}\}$. The nilpotent operator N_λ restricted to $\mathbb{K}^5/\operatorname{span}\{e_{12} + e_{21}, e_{11}\}$ has Jordan basis $\{e_{14}, e_{13}, e_{12}, e_{22}\}$ and Jordan type $(3,1)$. Therefore, $\lambda(v) = (3,1)$. Next, $N_\lambda v = e_{11}$ and $\lambda(N_\lambda v) = (3,2)$. Finally, $\lambda(N_\lambda^2 v) = (4,2)$ since $N_\lambda^2 v = 0$.*

Notice in the last example, we have the Young diagrams

$\lambda(v)$:  $\lambda(N_\lambda v)$:  $\lambda(N_\lambda^2 v)$: 

where $\lambda(v)$ and $\lambda(N_\lambda v)$ differ by exactly one box in column 2 , which equals $\mathrm{ht}_\lambda(v)$; also, $\lambda(N_\lambda v)$ and $\lambda(N_\lambda^2 v)$ differ by exactly one box in column 4, which equals $\mathrm{ht}_\lambda(Xv)$. We will show that this is not a coincidence in the next proposition. We need one lemma first:

Lemma 3.24. *If v is a nonzero vector in $\ker(N_\lambda)$, the Young diagrams of $\lambda(v)$ and λ differ by exactly one box, and one can obtain the diagram of $\lambda(v)$ by deleting a box at the end of column $\mathrm{ht}_\lambda(v)$ from the diagram of λ.*

Proof. Let $d = \mathrm{ht}_\lambda(v)$. Since $v \in \ker(N_\lambda)$, v is an eigenvector of N_λ. By the definition of height, every maximal Jordan chain containing v must have length d. Suppose one of them is in the following form:

$$\{w, \quad N_\lambda w, \quad N_\lambda^2 w, \quad \ldots, \quad N_\lambda^{d-1} w = v\}.$$

Then it can be expanded to become a new Jordan basis of N_λ. If we restrict *NTolambda* to the quotient space $\mathbb{K}^n/\langle v\rangle_\lambda$, the image of this Jordan basis in the quotient becomes a Jordan basis of $\mathbb{K}^n/\langle v\rangle_\lambda$ for the operator $N_\lambda|_{\mathbb{K}^n/\langle v\rangle_\lambda}$. Therefore, the Young diagrams of $\lambda(v)$ and λ differ by exactly one box in column d. □

Proposition 3.25. *For every nonzero vector $v \in \mathbb{K}^n$, the Young diagrams of $\lambda(v)$ and $\lambda(N_\lambda v)$ differ by exactly one box, and we obtain the diagram of $\lambda(v)$ by deleting a box at the end of column $\mathrm{ht}_\lambda(v)$ from the diagram of $\lambda(N_\lambda v)$.*

Proof. The case that $v \in \ker(N_\lambda)$ follows from Lemma 3.24. So it suffices to prove the statement for $v \notin \ker(N_\lambda)$.

Let $\overline{N_\lambda} = N_\lambda|_W$, where $W = \mathbb{K}^n/\langle N_\lambda v\rangle_\lambda$. We can choose a Jordan basis for this operator so that we can identify $\overline{N_\lambda}$ with N_μ, where $\mu = \lambda(N_\lambda v)$. Now, let $\overline{v}$ be the image of v under this quotient. By the third isomorphism theorem for modules, we have

$$\mathbb{K}^n/\langle v\rangle_\lambda \cong \left(\mathbb{K}^n/\langle N_\lambda v\rangle_\lambda\right) \;/\; \left(\langle v\rangle_\lambda/\langle N_\lambda v\rangle_\lambda\right) \cong W/(\langle \overline{v}\rangle_\mu)$$

So, we have $\lambda(v) = \mu(\overline{v})$. And since $\overline{v} \in \ker(N_\mu)$, by Lemma 3.24, the diagrams of $\mu(\overline{v})$ and μ differ by only one box in column $\mathrm{ht}_\mu(\overline{v})$. Therefore, it suffices to prove that $\mathrm{ht}_\mu(\overline{v}) = \mathrm{ht}_\lambda(v)$.

For simplicity, we let $\mathrm{ht}_\lambda(v) = d$. Then, there must exist some vector w such that $N_\lambda^{d-1}w = v$. If we let $\overline{w}$ be the image of w under the quotient map, we will have $N_\mu^{d-1}\overline{w} = \overline{v}$. So $\mathrm{ht}_\mu(\overline{v}) \geq d$.

Assume that, for the sake of contradiction, $\mathrm{ht}_\mu(\overline{v}) > d$; then there exists some vector $\overline{y}$ such that $N_\mu^d\overline{y} = \overline{v}$. Let y be a vector such that its image under the quotient is $\overline{y}$. Then, we have $N_\lambda^d y = v+u$ for some $u \in \langle N_\lambda v\rangle_\lambda$ and have $v+u \in \mathrm{im}(N_\lambda^d)$. By Lemma 3.14 and the shift property, we have $\mathrm{ht}_\lambda(u) \geq \mathrm{ht}_\lambda(N_\lambda v) \geq \mathrm{ht}_\lambda(v)+1 = d+1$, which implies $u \in \mathrm{im}(N_\lambda^d)$. Hence, we also have $v = (v+u) - u \in \mathrm{im}(N_\lambda^d)$, which contradicts the fact that $\mathrm{ht}_\lambda(v) = d$.

Therefore, we have $\mathrm{ht}_\mu(\overline{v}) = \mathrm{ht}_\lambda(v)$, and $\mathrm{ht}_\lambda(v)$ is the unique column index at which the diagrams of $\lambda(v)$ and $\lambda(N_\lambda v)$ differ. $\square$

Corollary 3.26. *For every nonzero vector $v \in \mathbb{K}^n$, we can obtain the Young diagram of $\lambda(v)$ by deleting a box at the end of the columns*

$$\mathrm{ht}_\lambda(v) < \mathrm{ht}_\lambda(N_\lambda v) < \mathrm{ht}_\lambda(N_\lambda^2 v) < \dots$$

from the diagram of λ.

Proof. By Proposition 3.25, we can obtain the diagram of $\lambda(v)$ by deleting a box at the end of column $\mathrm{ht}_\lambda(v)$ from the diagram of $\lambda(N_\lambda v)$. Then, by Proposition 3.25 again, we can obtain the diagram of $\lambda(v)$ by deleting a box at the end of columns $\mathrm{ht}_\lambda(v)$ and $\mathrm{ht}_\lambda(N_\lambda v)$ from the diagram of $\lambda(N_\lambda^2 v)$. Continuing this process, we can finally obtain the Young diagrams of $\lambda(v)$ by deleting a box at the end of the columns $\mathrm{ht}_\lambda(v),\ \mathrm{ht}_\lambda(N_\lambda v), \mathrm{ht}_\lambda(N_\lambda^2 v)\dots$ from the Young diagram of λ. $\square$

Below, we let λ be a partition of n.

Proposition 3.27. *For every partition μ, the set*

$$\{v \in \mathbb{K}^n : v \neq 0, \lambda(v) = \mu\}$$

is not empty if and only if λ/μ is a horizontal strip or, in other words, $\mu \in \Gamma(\lambda)$.

Proof. For every nonzero vector v, let $\mu = \lambda(v)$. Then, by Corollary 3.26, μ can be obtained by deleting one box at the end of the columns with index

$$\mathrm{ht}_\lambda(v), \mathrm{ht}_\lambda(N_\lambda v), \mathrm{ht}_\lambda(N_\lambda^2 v), \dots$$

from the Young diagram of λ. By Remark 3.16, all of these column indices are distinct from one another. Therefore, λ/μ must be a horizontal strip.

On the other hand, for every partition μ such that λ/μ is a horizontal strip, we claim that there exists a nonzero vector v such that $\lambda(v) = \mu$. Let $d = |\lambda| - |\mu|$, we prove the claim by induction on d.

Base case: When $d = 1$. Suppose ℓ is the unique column index such that ${\lambda_\ell}^* - \mu_\ell^* = 1$. Notice that we must have $\lambda_\ell^* > \lambda_{\ell+1}^*$; Otherwise, the diagram of μ, obtained by deleting a box at the end of column ℓ from the Young diagram of λ, cannot be a Young diagram. Then, by Corollary 3.7, there exists some v such that $v \in \ker(N_\lambda)$ and $\mathrm{ht}_\lambda(v) = \ell$, which, by Lemma 3.24, means we have $\lambda(v) = \mu$.

Inductive step: When $d > 1$, suppose that

$$\ell_1 < \quad \ell_2 < \quad \ldots\ldots \quad < \ell_{d-1} < \quad \ell_d$$

are the column indices such that $\lambda_i^* - \mu_i^* = 1$. Let ν be the partition that satisfies $\lambda_i^* - \nu_i^* = 1$ for $i \in \{\ell_2, \ell_3, \ldots, \ell_d\}$, and $\lambda_i^* = \nu_i^*$ otherwise. Then, λ/ν is a horizontal strip, and $|\lambda| - |\nu| = d - 1$. Inductively, assume that we have already proved that there exists a vector u such that $\lambda(u) = \nu$. By Corollary 3.26, we have $\mathrm{ht}_\lambda(N_\lambda^{i-2}u) = \ell_i$ for $i \in \{2, 3, \ldots, d\}$. There are two possible cases,

(1) $\ell_1 = \ell_2 - 1$. By Lemma 3.9, there always exists some v such that $N_\lambda v = u$ and $\mathrm{ht}_\lambda(v) = \ell_2 - 1 = \ell_1$.
(2) $\ell_1 < \ell_2 - 1$. Notice that we must have $\lambda_{\ell_1}^* > \lambda_{\ell_1+1}^*$; Otherwise, the diagram of μ, obtained by deleting a box at the end of column ℓ_i where $i \in [d]$ from the Young diagram of λ, cannot be a Young diagram. Then, also by Lemma 3.9, we can always choose some v such that $N_\lambda v = u$ and $\mathrm{ht}_\lambda(v) = \ell_1$.

In both cases, we have $\lambda(v) = \mu$, since $\mathrm{ht}_\lambda(v) = \ell_1$, and $\mathrm{ht}_\lambda(N_\lambda^{i-1}v) = \mathrm{ht}_\lambda(N_\lambda^{i-2}u) = \ell_i$ for $i \in \{2, 3, \ldots, d\}$. □

Recall that when λ/μ is a horizontal strip, we define $I_{\lambda/\mu}$ as the set of all column indices j such that $\lambda_j^* - \mu_j^* = 1$. Then, by Corollary 3.26 and the previous proposition, there always exists some vector v such that

$$I_{\lambda/\mu} = \left\{\mathrm{ht}_\lambda(v), \mathrm{ht}_\lambda(N_\lambda v), \mathrm{ht}_\lambda(N_\lambda^2 v), \ldots\right\}. \tag{3.2}$$

Example 3.28. *If $\lambda = (3, 2, 1)$ and $\mu = (2, 2)$, then λ/μ is a horizontal strip, and $I_{\lambda/\mu} = \{1, 3\}$. By Corollary 3.26, for every vector v, $\lambda(v) = \mu$ if and only if $N_\lambda v \in \ker(N_\lambda)$, $\mathrm{ht}_\lambda(v) = 1$, and $\mathrm{ht}_\lambda(N_\lambda v) = 3$. Then, we can let v be any vector of the form $c_{31}e_{31} + c_{21}e_{21} + c_{11}e_{11} + c_{12}e_{12}$ such that $c_{31} \neq 0$, $c_{12} \neq 0$, and other coefficients can be any number in $\mathbb{K}$. In this case, we have $\mathrm{ht}_\lambda(v) = 1$, $\mathrm{ht}_\lambda(N_\lambda v) = 3$, and $\lambda(v) = (2, 2)$.*

In the next proposition, we count how many vectors v there are in $\mathbb{F}_q^n$ for a fixed horizontal strip λ/μ that satisfy the Formula (3.2).

Proposition 3.29. *Given partitions λ and μ such that λ/μ is a horizontal strip, we have the equation*

$$\#\{v \in \mathbb{F}_q^n : \lambda(v) = \mu\} = \prod_{i \in I_{\lambda/\mu}} g_i(q), \tag{3.3}$$

where $g_i(q) := \begin{cases} q^{\lambda_i^*}, & \text{if } i + 1 \in I_{\lambda/\mu} \\ q^{\lambda_i^*} - q^{\lambda_{i+1}^*}, & \text{otherwise.} \end{cases}$

Proof. Let $d = |\lambda| - |\mu|$. We prove the statement using induction on d.

Base case: When $d = 1$, suppose $I_{\lambda/\mu} = \{\ell\}$. By Corollary 3.26, for every nonzero vector v, $\lambda(v) = \mu$ if and only if $v \in \ker(N_\lambda)$ and $\mathrm{ht}_\lambda(v) = \ell$. And by Corollary 3.7, there are $q^{\lambda_\ell^*} - q^{\lambda_{\ell+1}^*}$ $v \in \ker(N_\lambda)$ that satisfy $\mathrm{ht}_\lambda(v) = \ell$.

Inductive step: When $d > 1$, suppose that $I_{\lambda,\mu} = \{\ell_1, \ell_2, \ldots, \ell_d\}$ such that $\ell_1 < \ell_2 < \ldots < \ell_{d-1} < \ell_d$. Let ν be the partition that satisfies $\lambda_i^* - \nu_i^* = 1$ for $i \in \{\ell_2, \ell_3, \ldots, \ell_d\}$, and $\lambda_i^* = \nu_i^*$ otherwise. Then, λ/ν is a horizontal strip, and $|\lambda| - |\nu| = d - 1$.

By Corollary 3.26, for every vector $v \in \mathbb{K}^n$, $\lambda(v) = \mu$ if and only if $N_\lambda^{d-1} v \in \ker(N_\lambda)$ and

$$\mathrm{ht}_\lambda(v) = \ell_1,\ \mathrm{ht}_\lambda(N_\lambda v) = \ell_2,\ \mathrm{ht}_\lambda(N_\lambda^2 v) = \ell_3, \dots,\ \mathrm{ht}(N_\lambda^{d-1} v) = \ell_d.$$

And by Corollary 3.26 again, for every vector $v \in \mathbb{K}^n$, the condition $N_\lambda^{d-1} v \in \ker(N_\lambda)$ and

$$\mathrm{ht}_\lambda(N_\lambda v) = \ell_2,\ \mathrm{ht}_\lambda(N_\lambda^2 v) = \ell_3, \dots,\ \mathrm{ht}(N_\lambda^{d-1} v) = \ell_d.$$

is equivalent to $\lambda(N_\lambda v) = \nu$. Therefore, for every vector v, $\lambda(v) = \mu$ if and only if $\mathrm{ht}_\lambda(v) = d_1$ and $\lambda(N_\lambda v) = \nu$. By induction, the cardinality of the set

$$\left\{u \in \mathbb{F}_q^n : \lambda(u) = \nu\right\}$$

is equal to

$$G(q) := \prod_{i \in I_{\lambda/\nu}} g_i(q), \text{ where } g_i(q) := \begin{cases} q^{\lambda_i^*}, & \text{if } i+1 \in I_{\lambda/\nu} \\ q^{\lambda_i^*} - q^{\lambda_{i+1}^*}, & \text{otherwise.} \end{cases} \tag{3.4}$$

There are two possible cases,

(1) $\ell_1 = \ell_2 - 1$. By Lemma 3.9, for a fixed u in the set $\{u \in \mathbb{F}_q^n : \lambda(u) = \nu\}$, the number of vectors v such that $N_\lambda v = u$ and $\mathrm{ht}_\lambda(v) = \ell_2 - 1 = \ell_1$ is equal to $q^{\lambda_{\ell_1}^*}$. Notice that we have $\ell_2 = \ell_1 + 1 \in I_{\lambda/\mu}$, the number of v such that $\lambda(v) = \mu$ is given by the formula

$$q^{\lambda_{\ell_1}^*} \cdot \#\{u \in \mathbb{F}_q^n : \lambda(u) = \nu\} = q^{\lambda_{\ell_1}^*} \cdot G(q) = \prod_{i \in I_{\lambda/\mu}} g_i(q)$$

where $g_i(q) = \begin{cases} q^{\lambda_i^*}, & \text{if } i+1 \in I_{\lambda/\mu} \\ q^{\lambda_i^*} - q^{\lambda_{i+1}^*}, & \text{otherwise.} \end{cases}$

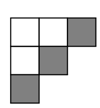
(2) $\ell_1 < \ell_2 - 1$. By Lemma 3.9, for a fixed u in the set $\{u \in \mathbb{F}_q^n : \lambda(u) = \nu\}$, the number of vectors v such that $N_\lambda v = u$ and $\mathrm{ht}_\lambda(v) = \ell_1$ is equal to $q^{\lambda_{\ell_1}^*} - q^{\lambda_{\ell_1+1}^*}$. Notice that we have $\ell_1 + 1 \notin I_{\lambda/\mu}$. Thus, the number of v such that $\lambda(v) = \mu$ is given by the formula

$$(q^{\lambda_{\ell_1}^*} - q^{\lambda_{\ell_1+1}^*}) \cdot \#\{u \in \mathbb{F}_q^n : \lambda(u) = \nu\} = (q^{\lambda_{\ell_1}^*} - q^{\lambda_{\ell_1+1}^*}) \cdot G(q) = \prod_{i \in I_{\lambda/\mu}} g_i(q)$$

where $g_i(q) = \begin{cases} q^{\lambda_i^*}, & \text{if } i+1 \in I_{\lambda/\mu} \\ q^{\lambda_i^*} - q^{\lambda_{i+1}^*}, & \text{otherwise.} \end{cases}$

In both cases, Formula (3.3) gives the number of $v \in \mathbb{F}_q^n$ such that $\lambda(v) = \mu$. □

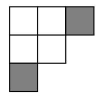
Example 3.30. *Let $\lambda = (3,2,1)$ and $\mu = (2,1)$. Then, $\lambda^* = (3,2,1)$ and λ/μ is the following horizontal strip (gray part).*



Therefore, $I_{(3,2,1)/(2,1)} = \{1,2,3\}$ and by Proposition 3.29, we have

$$\#\{v \in \mathbb{F}_q^6 : \lambda(v) = (2,1)\} = q^3 \cdot q^2 \cdot (q-1) = q^5(q-1).$$

Example 3.31. *Let* $\lambda = (3, 2, 1)$ *and* $\mu = (2, 2)$*. Then,* $\lambda^* = (3, 2, 1)$ *and* λ/μ *is the following horizontal strip (gray part).*



Therefore, $I_{(3,2,1)/(2,2)} = \{1, 3\}$ *and by Proposition 3.29, we have*

$$\#\{v \in \mathbb{F}_q^6 : \lambda(v) = (2, 2)\} = (q^3 - q^2) \cdot (q - 1) = q^2(q - 1)^2.$$

Corollary 3.32. *Let* λ *and* μ *be partitions such that* $|\lambda| = n$*, and* λ/μ *is a horizontal strip. Then, the cardinality of the set*

$$\{v \in \mathbb{F}_q^n : \lambda(v) = \mu\}$$

is a polynomial in q *whose leading term is* $q^{\sum_{i\in[\ell(\lambda)]} i(\lambda_i - \mu_i)}$*, and it is always divisible by* $q - 1$*.*

Proof. By Proposition 3.29, the number of vectors in $\{v \in \mathbb{F}_q^n : \lambda(v) = \mu\}$ equals

$$G(q) := \prod_{i\in I_{\lambda/\mu}} g_i(q), \text{ where } g_i(q) := \begin{cases} q^{\lambda_i^*}, & \text{if } i + 1 \in I_{\lambda/\mu} \\ q^{\lambda_i^*} - q^{\lambda_{i+1}^*}, & \text{otherwise.} \end{cases}$$

Thus, $G(q)$ is a polynomial in q whose leading term equals

$$\prod_{j\in I_{\lambda/\mu}} q^{\lambda_j^*} = q^{\sum_{j\in I_{\lambda/\mu}} \lambda_j^*}.$$

We claim that there is an equation $\sum_{j\in I_{\lambda/\mu}} \lambda_j^* = \sum_{i\in[k]} i(\lambda_i - \mu_i)$. For every $i \in [\ell(\lambda)]$, we have

$$\#\{j \in I_{\lambda/\mu} : \lambda_j^* = i\} = \#\{\text{Boxes in the } i\text{-th row of the skew Young diagram } \lambda/\mu\} = \lambda_i - \mu_i,$$

And hence,

$$\sum_{j\in I_{\lambda/\mu}} \lambda_j^* = \sum_{i\in[\ell(\lambda)]} i \cdot \#\{j \in I_{\lambda/\mu} : \lambda_j^* = i\} = \sum_{i\in[\ell(\lambda)]} i(\lambda_i - \mu_i)$$

which is the degree of the polynomial $G(q)$.

To check that $G(q)$ is always divisible by $q - 1$, let t be the maximal integer in $I_{\lambda/\mu}$. Then, $G(q)$ has a factor $g_t(q) = q^{\lambda_t^*} - q^{\lambda_{t+1}^*}$ that can be divided by $q - 1$. $\square$

4. Admissible decomposition

In this section, we discuss how to decompose the generalized parabolic Peterson variety $\mathrm{Pet}_{\lambda,\alpha}$ into a union of $\mathrm{Pet}_{\lambda,\beta}$ such that no variety in the union is contained in any other. Here we assume the underlying field is $\mathbb{C}$. However, all the proofs below also hold for the finite field $\mathbb{F}_q$, since we do not require the characteristic to be 0.

Recall that for every partition λ of n, we let N_λ be the nilpotent matrix in Jordan form of Jordan type λ, and $(e_{ij})_\lambda$ be an orthogonal Jordan basis of N_λ. Recall that $\mathcal{D}_\lambda$ is the set of all compositions β of n such that $\beta \trianglelefteq \lambda$. For every composition α of n, $\mathcal{R}_\alpha$ is the set of all compositions β of n such that $\beta \preceq \alpha$. We say that $\mathrm{Pet}_{\lambda,\alpha}$ is **admissible** if $\alpha \trianglelefteq \lambda$, or in other words, $\alpha \in \mathcal{D}_\lambda$. Notice that the condition $\alpha \trianglelefteq \lambda$ is also equivalent to the condition that the Kostka number $\mathcal{K}_{\lambda\alpha}$ is positive. For compositions α and β such that $|\alpha| = |\beta|$, recall that $\alpha \circ \beta$ is the composition such that $PS(\alpha \circ \beta) = PS(\alpha) \cup PS(\beta)$.

4.1. **The composition associated with a flag.** For a flag $F_\bullet \in \mathrm{Fl}(n)$, we say that $F_\bullet$ is determined by an ordered basis

$$(b_1, \quad b_2, \quad \ldots\ldots, \quad b_{n-1}, \quad b_n),$$

if $F_i = \mathrm{span}\{b_1, b_2, \ldots, b_i\}$ for all $i \in [n]$.

Definition 4.1. *Fix a partition λ of n. For any $F_\bullet \in \mathrm{Pet}_\lambda$, we let $\alpha_\lambda(F_\bullet)$ be the unique composition such that for all $i \in [n]$,*

$$N_\lambda(F_i) \subset F_i \text{ if and only if } i \in PS(\alpha_\lambda(F_\bullet)).$$

By definition, for every partition λ and composition α such that $|\lambda| = |\alpha|$, a flag $F_\bullet \in \mathrm{Pet}_\lambda$ is in $\mathrm{Pet}_{\lambda,\alpha}$ if and only if $\alpha_\lambda(F_\bullet)$ refines α.

Example 4.2. *Let $\lambda = (3,1)$ and $\alpha = (2,2)$, suppose $F_\bullet$ is the flag determined by the following ordered basis*

$$(e_{12} \;+\; e_{21}, \quad e_{11}, \quad e_{13}, \quad e_{21}).$$

Then, we have

$$\begin{aligned}
N_\lambda(F_1) &= \mathrm{span}\{N_\lambda(e_{12}+e_{21})\} = \mathrm{span}\{e_{11}\},\\
N_\lambda(F_2) &= \mathrm{span}\{N_\lambda(e_{12}+e_{21}), N_\lambda(e_{11})\} = \mathrm{span}\{e_{11}\},\\
N_\lambda(F_3) &= \mathrm{span}\{N_\lambda(e_{12}+e_{21}), N_\lambda(e_{11}), N_\lambda(e_{13})\} = \mathrm{span}\{e_{11}, e_{12}\}.
\end{aligned}$$

Therefore, we have $N_\lambda(F_1) \not\subseteq F_1$ while $N_\lambda(F_1) \subseteq F_2$, $N_\lambda(F_2) \subseteq F_2 \subseteq F_3$, $N_\lambda(F_3) \not\subseteq F_3$ while $N_\lambda(F_3) \subseteq F_4$. We conclude that $F_\bullet \in \mathrm{Pet}_{(3,1)}$ and $\alpha_\lambda(F_\bullet) = (2,2)$. Also, $F_\bullet \in \mathrm{Pet}_{(3,1),(2,2)}$.

Proposition 4.3. *Given partition λ and compositions α and β such that $|\lambda| = |\alpha| = |\beta|$, we have*

1. $\mathrm{Pet}_{\lambda,\alpha} \subseteq \mathrm{Pet}_{\lambda,\beta}$ *if α is a refinement of β.*
2. $\mathrm{Pet}_{\lambda,\alpha} \cap \mathrm{Pet}_{\lambda,\beta} = \mathrm{Pet}_{\lambda,\gamma}$ *, where $\gamma = \alpha \circ \beta$.*

Proof. To show (1), for every $F_\bullet \in \mathrm{Pet}_{\lambda,\alpha}$, we have $\alpha_\lambda(F_\bullet)$ refines α. By assumption, α refines β, so $\alpha_\lambda(F_\bullet)$ refines β too. Therefore, $F_\bullet \in \mathrm{Pet}_{\lambda,\beta}$ and $\mathrm{Pet}_{\lambda,\alpha} \subseteq \mathrm{Pet}_{\lambda,\beta}$.

To show (2), since γ refines both α and β, by (1) we have $\mathrm{Pet}_{\lambda,\gamma} \subseteq \mathrm{Pet}_{\lambda,\alpha}$ and $\mathrm{Pet}_{\lambda,\gamma} \subseteq \mathrm{Pet}_{\lambda,\beta}$. Therefore, $\mathrm{Pet}_{\lambda,\gamma} \subseteq \mathrm{Pet}_{\lambda,\alpha} \cap \mathrm{Pet}_{\lambda,\beta}$. On the other hand, if $F_\bullet \in \mathrm{Pet}_{\lambda,\alpha} \cap \mathrm{Pet}_{\lambda,\beta}$, we have that $\alpha_\lambda(F_\bullet)$ refines both α and β and hence it refines $\alpha \circ \beta$. Therefore, $F_\bullet \in \mathrm{Pet}_{\lambda,\gamma}$ and $\mathrm{Pet}_{\lambda,\alpha} \cap \mathrm{Pet}_{\lambda,\beta} \subseteq \mathrm{Pet}_{\lambda,\gamma}$. ☐

Remark 4.4. *Statements (1) and (2) of Proposition 4.3 follow from more general facts about Hessenberg varieties. For an arbitrary complex matrix X and two Hessenberg functions h_1 and h_2, we have the following*

1. *If $h_1(i) \leq h_2(i)$ holds for all i, then* $\mathrm{Hess}(X, h_1) \subset \mathrm{Hess}(X, h_2)$*, and*
2. $\mathrm{Hess}(X, h_1) \cap \mathrm{Hess}(X, h_2) = \mathrm{Hess}(X, h_3)$*, where $h_3 : \ i \mapsto \min\{h_1(i), h_2(i)\}$.*

In general, the converse statement of (1) of Proposition 4.3 is not true. For a counterexample, consider $\lambda = (1,1,1), \alpha = (2,1)$ and $\beta = (2,1)$. Note that $\mathrm{Pet}_{(1,1,1),(2,1)} = \mathrm{Pet}_{(1,1,1),(1,2)}$, since both varieties are equal to $\mathrm{Fl}(3)$. But neither $(2,1)$ nor $(1,2)$ is a refinement of the other. However, we show below that the converse statement will hold under the assumption that α and β are both in $\mathcal{D}_\lambda$. This fact is important to our study of the parabolic Peterson variety.

Proposition 4.5. *Given a partition λ and a composition α such that $|\lambda| = |\alpha|$, the set*

$$\{F_\bullet \in \mathrm{Pet}_\lambda : \alpha_\lambda(F_\bullet) = \alpha\}$$

is not empty if and only if $\alpha \trianglelefteq \lambda$, i.e., $\alpha \in \mathcal{D}_\lambda$.

We first discuss the important corollaries of this proposition and provide the proof in the next subsection.

Corollary 4.6. *For every partition λ and composition α such that $|\lambda| = |\alpha|$, we have*

(1) $\mathrm{Pet}_\lambda = \bigcup_{\beta \in \mathcal{D}_\lambda} \mathrm{Pet}_{\lambda,\beta}$.
(2) $\mathrm{Pet}_{\lambda,\alpha} = \cup_{\beta \in \mathcal{D}_\lambda \cap \mathcal{R}_\alpha} \mathrm{Pet}_{\lambda,\beta}$.

Proof. It suffices to prove (2) since (1) is a special case of (2) by substituting $\alpha = (n)$. To show (2) is true, for every $\beta \in \mathcal{D}_\lambda$ such that $\beta \preceq \alpha$, by Proposition 4.3, we have $\mathrm{Pet}_{\lambda,\beta} \subset \mathrm{Pet}_{\lambda,\alpha}$. Therefore, $\bigcup_{\beta \in \mathcal{D}_\lambda \cap \mathcal{R}_\alpha} \mathrm{Pet}_{\lambda,\beta} \subset \mathrm{Pet}_{\lambda,\alpha}$.

On the other hand, for every $F_\bullet \in \mathrm{Pet}_{\lambda,\alpha}$, let $\gamma = \alpha_\lambda(F_\bullet)$, so $F_\bullet \in \mathrm{Pet}_{\lambda,\gamma}$. By Proposition 4.5, we have $\gamma \trianglelefteq \lambda$ and we conclude

$$F_\bullet \in \mathrm{Pet}_{\lambda,\gamma} \subseteq \cup_{\beta \in \mathcal{D}_\lambda \cap \mathcal{R}_\alpha} \mathrm{Pet}_{\lambda,\beta}\,. \qquad \square$$

Corollary 4.7. *Given a partition λ and compositions α and β such that $\alpha, \beta \in \mathcal{D}_\lambda$, $\mathrm{Pet}_{\lambda,\alpha} \subseteq \mathrm{Pet}_{\lambda,\beta}$ if and only if α is a refinement of β.*

Proof. When $\alpha \preceq \beta$, we have $\mathrm{Pet}_{\lambda,\alpha} \subseteq \mathrm{Pet}_{\lambda,\beta}$ by Proposition 4.3. On the other hand, suppose $\alpha, \beta \in \mathcal{D}_\lambda$ and $\alpha \not\preceq \beta$. By Proposition 4.5, there exists a flag $F_\bullet$ such that $\alpha_\lambda(F_\bullet) = \alpha$ which is not a refinement of β. Therefore, $F_\bullet \in \mathrm{Pet}_{\lambda,\alpha}$ while $F_\bullet \notin \mathrm{Pet}_{\lambda,\beta}$. So, we have $\mathrm{Pet}_{\lambda,\alpha} \not\subseteq \mathrm{Pet}_{\lambda,\beta}$ if $\alpha \not\preceq \beta$, which finishes the proof. $\square$

Remark 4.8. *By Proposition 4.3 and Corollary 4.7, for every partition λ there is an isomorphism*

$$(\mathcal{D}_\lambda,\ \preceq,\ \circ) \cong (\{\mathrm{Pet}_{\lambda,\alpha} : \alpha \in \mathcal{D}_\lambda\},\ \subseteq,\ \cap)$$

of posets defined by $\alpha \mapsto \mathrm{Pet}_{\lambda,\alpha}$ that preserves the relevant binary operations.

Example 4.9. *Let $\lambda = (3,1)$. By Corollary 4.6, we have*

$$\begin{aligned}\mathrm{Pet}_{(3,1)} = &\mathrm{Pet}_{(3,1),(3,1)} \cup \mathrm{Pet}_{(3,1),(2,2)} \cup \mathrm{Pet}_{(3,1),(1,3)} \\ &\cup \mathrm{Pet}_{(3,1),(2,1,1)} \cup \mathrm{Pet}_{(3,1),(1,2,1)} \cup \mathrm{Pet}_{(3,1),(1,1,2)} \cup \mathrm{Pet}_{(3,1),(1,1,1,1)}\,.\end{aligned}$$

By Corollary 4.7, we further have the decomposition

$$\mathrm{Pet}_{(3,1)} = \mathrm{Pet}_{(3,1),(3,1)} \cup \mathrm{Pet}_{(3,1),(2,2)} \cup \mathrm{Pet}_{(3,1),(1,3)}$$

satisfies the condition that none of $\mathrm{Pet}_{(3,1),(3,1)}$, $\mathrm{Pet}_{(3,1),(2,2)}$ or $\mathrm{Pet}_{(3,1),(1,3)}$ is contained in the other. The Hasse diagram of the poset $(\{\mathrm{Pet}_{\lambda,\alpha} : \alpha \in \mathcal{D}_\lambda\}, \subseteq)$ is the *following.*

$$\begin{tikzcd}[column sep=small]
\mathrm{Pet}_{(3,1),(3,1)} \arrow[d, dash] \arrow[dr, dash] & \mathrm{Pet}_{(3,1),(2,2)} \arrow[dl, dash] \arrow[dr, dash] & \mathrm{Pet}_{(3,1),(1,3)} \arrow[dl, dash] \arrow[d, dash] \\
\mathrm{Pet}_{(3,1),(2,1,1)} \arrow[dr, dash] & \mathrm{Pet}_{(3,1),(1,2,1)} \arrow[d, dash] & \mathrm{Pet}_{(3,1),(1,1,2)} \arrow[dl, dash] \\
& \mathrm{Pet}_{(3,1),(1,1,1,1)} = \mathcal{B}_{(3,1)} &
\end{tikzcd}$$

The reader can confirm that this diagram is the same as the one for $(\mathcal{D}_\lambda, \preceq)$ *in Example 2.5.*

Example 4.10. *Let* $\lambda = (2,2)$. *By Corollary 4.6, we have*

$$\begin{aligned}\mathrm{Pet}_{(2,2)} = &\mathrm{Pet}_{(2,2),(2,2)} \cup \mathrm{Pet}_{(2,2),(1,2,1)} \\ &\cup \mathrm{Pet}_{(2,2),(2,1,1)} \cup \mathrm{Pet}_{(2,2),(1,1,2)} \cup \mathrm{Pet}_{(2,2),(1,1,1,1)}\,.\end{aligned}$$

By Corollary 4.7, we further have the decomposition

$$\mathrm{Pet}_{(2,2)} = \mathrm{Pet}_{(2,2),(2,2)} \cup \mathrm{Pet}_{(2,2),(1,2,1)}$$

such that neither of $\mathrm{Pet}_{(2,2),(2,2)}$ *or* $\mathrm{Pet}_{(2,2),(1,2,1)}$ *is contained in the other. The Hasse diagram of the poset* $(\{\mathrm{Pet}_{\lambda,\alpha} : \alpha \in \mathcal{D}_\lambda\}, \subseteq)$ *as the following.*

$$\begin{tikzcd}[column sep=tiny]
& & & \mathrm{Pet}_{(2,2),(2,2)} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{Pet}_{(2,2),(1,2,1)} \arrow[drrr, dash] & & \mathrm{Pet}_{(2,2),(2,1,1)} \arrow[dr, dash] & & \mathrm{Pet}_{(2,2),(1,1,2)} \arrow[dl, dash] \\
& & & \mathrm{Pet}_{(2,2),(1,1,1,1)} = \mathcal{B}_{(2,2)} &
\end{tikzcd}$$

The reader can confirm that this diagram is the same as the one for $(\mathcal{D}_\lambda, \preceq)$ *in Example 2.6.*

Theorem 1.2 states that for every $\mathrm{Pet}_{\lambda,\alpha}$, we have the decomposition

$$\mathrm{Pet}_{\lambda,\alpha} = \bigcup_{\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \mathrm{Pet}_{\lambda,\beta} \tag{4.1}$$

such that no variety in the union is contained within another. From now on, we will call this the **admissible decomposition** of $\mathrm{Pet}_{\lambda,\alpha}$. We are ready to provide proof of this theorem.

Proof of Theorem 1.2. For every β such that $\beta \preceq \alpha$ and $\beta \trianglelefteq \lambda$, we have $\mathrm{Pet}_{\lambda,\beta} \subset \mathrm{Pet}_{\lambda,\alpha}$ by Proposition 4.3. Therefore,

$$\bigcup_{\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \mathrm{Pet}_{\lambda,\beta} \subseteq \mathrm{Pet}_{\lambda,\alpha}\,. \tag{4.2}$$

To show the opposed inclusion, by Corollary 4.6 we have

$$\mathrm{Pet}_{\lambda,\alpha} = \bigcup_{\beta \in \mathcal{D}_\lambda \cap \mathcal{R}_\alpha} \mathrm{Pet}_{\lambda,\beta}\,.$$

Every β in the set $\mathcal{D}_\lambda \cap \mathcal{R}_\alpha$ refines at least one of the coarsest elements, which may be β itself. By Corollary 4.7, we have $\mathrm{Pet}_{\lambda,\beta} \subseteq \bigcup_{\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \mathrm{Pet}_{\lambda,\beta}$ and so

$$\mathrm{Pet}_{\lambda,\alpha} = \bigcup_{\beta \in \mathcal{D}_\lambda \cap \mathcal{R}_\alpha} \mathrm{Pet}_{\lambda,\beta} \subseteq \bigcup_{\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \mathrm{Pet}_{\lambda,\beta}\,.$$

Therefore, combining this with (4.2), we have

$$\mathrm{Pet}_{\lambda,\alpha} = \bigcup_{\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \mathrm{Pet}_{\lambda,\beta}\,. \tag{4.3}$$

Notice on the right-hand side, since all these β are coarsest, none is a refinement of any other. By Corollary 4.7 again, every variety $\mathrm{Pet}_{\lambda,\beta}$ in this union is not contained in any other. $\square$

Example 4.11. *When $\lambda = (4,2)$ and $\alpha = (5,1)$. The set $\left(\mathcal{D}_{(4,2)} \cap \mathcal{R}_{(5,1)}\right)^{\mathrm{crs}}$ is*

$$\big\{(4,1,1), (3,2,1), (2,3,1), (1,4,1)\big\}.$$

Therefore, by Theorem 1.2, we have

$$\mathrm{Pet}_{(4,2),(5,1)} = \mathrm{Pet}_{(4,2),(4,1,1)} \cup \mathrm{Pet}_{(4,2),(3,2,1)} \cup \mathrm{Pet}_{(4,2),(2,3,1)} \cup \mathrm{Pet}_{(4,2),(1,4,1)}\,.$$

4.2. **The proof of Proposition 4.5.** We begin with a technical lemma.

Lemma 4.12. *Let λ be a partition and α be a composition such that $|\lambda| = |\alpha|$ and $\ell(\alpha) = \ell$. If $F_\bullet$ is a flag in Pet_λ such that $\alpha_\lambda(F_\bullet) = \alpha$, then $F_\bullet$ is determined by an ordered basis of the form*

$$\left(v_1,\ N_\lambda v_1,\ \ldots,\ {N_\lambda}^{\alpha_1 - 1} v_1,\ v_2,\ N_\lambda v_2,\ \ldots,\ {N_\lambda}^{\alpha_2-1} v_2,\ \ldots,\ v_\ell,\ N_\lambda v_\ell,\ \ldots, {N_\lambda}^{\alpha_\ell - 1} v_\ell\right),$$

for vectors $v_1, v_2, \ldots, v_\ell \in \mathbb{C}^n$.

Proof. Let $v_1 \in \mathbb{C}^n$ be a vector that spans F_1. We will prove by induction that for all $i \leq \alpha_1$, we have

$$F_i = \mathrm{span}\{v_1, N_\lambda v_1, N_\lambda^2 v_1, \ldots, N_\lambda^{i-1} v_1\}.$$

The case $i = 1$ holds by assumption. When $i > 1$, assume that

$$F_{i-1} = \mathrm{span}\{v_1, N_\lambda v_1, N_\lambda^2 v_1, \ldots, N_\lambda^{i-2} v_1\}.$$

Then,

$$N_\lambda F_{i-1} = \mathrm{span}\{N_\lambda v_1, N_\lambda^2 v_1, N_\lambda^3 v_1, \ldots, N_\lambda^{i-1} v_1\}.$$

Since $i - 1 < \alpha_1$, by the definition of the composition $\alpha = \alpha_\lambda(F_\bullet)$, we have $N_\lambda v_1 \subseteq F_i$ while $N_\lambda v_1 \nsubseteq F_{i-1}$. Hence, $F_i = \mathrm{span}\{v_1, N_\lambda v_1, N_\lambda^2 v_1, \ldots, N_\lambda^{i-1} v_1\}$ for all $i \leq \alpha_1$. Specifically, $F_{\alpha_1} = \mathrm{span}\{v_1, N_\lambda v_1, N_\lambda^2 v_1, \ldots, N_\lambda^{\alpha_1 - 1} v_1\}$.

Then, since $F_{\alpha_1} \subseteq F_{\alpha_1 + 1}$, the subspace F_{α_1+1} must be of the form

$$\mathrm{span}\{v_1, N_\lambda v_1, N_\lambda^2 v_1, \ldots, N_\lambda^{\alpha_1 - 1} v_1, v_2\}$$

for some vector v_2. We can perform the same process and show that for all s $i < \alpha_2$, $F_{\alpha_1 + i}$ is of the form

$$\mathrm{span}\{v_1, N_\lambda v_1, N_\lambda^2 v_1, \ldots, N_\lambda^{\alpha_1 - 1} v_1, v_2, N_\lambda v_2, \ldots, N_\lambda^{i-1} v_2\}.$$

Specifically,

$$F_{\alpha_1+\alpha_2} = \mathrm{span}\{v_1, N_\lambda v_1, N_\lambda^2 v_1, \ldots, N_\lambda^{\alpha_1 - 1} v_1, v_2, N_\lambda v_2, \ldots, N_\lambda^{\alpha_2 - 1} v_2\}.$$

Continuing this process, we obtain the desired property. $\square$

Remark 4.13. *Given $F_\bullet \in \mathrm{Pet}_\lambda$, let $\alpha = \alpha_\lambda(F_\bullet)$, and suppose $v_1 \in \mathbb{C}^n$ is a vector that spans F_1. By the proof above, $N_\lambda^{\alpha_1-1}v \neq 0$ while $N_\lambda^{\alpha_1}v = 0$. Then, by Proposition 3.13, we have $\alpha_1 = \dim\langle v\rangle_\lambda$.*

Proof of Proposition 4.5. We first show the necessary part of the statement: for every $F_\bullet \in \mathrm{Pet}_\lambda$, let $\mu = (\mu_1, \ldots, \mu_\ell)$ be the partition obtained by rearranging the entries of $\alpha_\lambda(F_\bullet)$. We want to show that $\mu \trianglelefteq \lambda$, or, more explicitly, the following inequality that holds for every $j \in [\ell(\lambda)]$:

$$\mu_1 + \mu_2 + \ldots + \mu_j = \max_{\{i_1,\ldots,i_j\}\subset[\ell]} \left\{\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_j}\right\} \leq \lambda_1 + \lambda_2 + \ldots + \lambda_j. \tag{4.4}$$

By Lemma 4.12, for every $\{i_1, \ldots, i_j\} \subset [\ell]$, there exist $\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_j}$ different vectors of the form

$$v_{i_1}, N_\lambda v_{i_1}, \ldots, N_\lambda^{\alpha_{i_1}-1}v_{i_1}, v_{i_2}, N_\lambda v_{i_2}, \ldots, N_\lambda^{\alpha_{i_2}-1}v_{i_2}, \ldots v_{i_j}, N_\lambda v_{i_j}, \ldots, N_\lambda^{\alpha_{i_j}-1}v_{i_j}.$$

Notice that these $\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_j}$ vectors form at most j different λ-Jordan chains. But the longest j different λ-Jordan chains are of the length $\lambda_1, \lambda_2, \ldots \lambda_j$. Therefore, we have

$$\alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_j} \leq \lambda_1 + \lambda_2 + \ldots + \lambda_j.$$

Thus, Formula (4.4) is true, and we have proven the necessary part of Proposition 4.5.

To show the sufficient part of the statement, we show that for every partition λ and composition α such that $\alpha \trianglelefteq \lambda$, there exists at least one flag $F_\bullet \in \mathrm{Pet}_\lambda$ such that $\alpha_\lambda(F_\bullet) = \alpha$. We proceed by induction on $\ell := \ell(\alpha)$.

Base case: When $\ell = 1$, we have $\alpha = (n)$ for some n. By assumption $(n) \trianglelefteq \lambda$, so we must have $\lambda = \alpha = (n)$ and N_λ is the regular nilpotent matrix $N_{(n)}$. Consider the flag $F_\bullet$ determined by the ordered basis

$$\left(e_{1n}, \quad e_{1(n-1)}, \quad \ldots\ldots, \quad e_{12}, \quad e_{11}\right).$$

For all $i \in \{2, 3, \ldots, n\}$, since $N_{(n)}e_{1i} = e_{1(i-1)}$, we have

$$N_{(n)}F_i = \mathrm{span}\{e_{1(n-1)}, e_{1(n-2)}, \ldots, e_{1(n-i+1)}, e_{1(n-i)}\}.$$

Hence, for all $i \in [n-1]$,

$$N_{(n)}F_i \not\subset F_i = \mathrm{span}\{e_{1(n)}, \ldots, e_{1(n-i+2)}, e_{1(n-i+1)}\},$$

while

$$N_{(n)}F_i \subset F_{i+1} = \mathrm{span}\{e_{1(n)}, \ldots, e_{1(n-i+1)}, e_{1(n-i)}\}.$$

And trivially, $N_{(n)}F_n \subset F_n = \mathbb{C}^n$.

Inductive step: Suppose $\ell \geq 2$, and let $\overline{\alpha} = (\alpha_2, \alpha_3, \ldots, \alpha_\ell)$ be the composition obtained by deleting the first entry from α as defined in Section 2. Since $\alpha \trianglelefteq \lambda$, we have $\mathcal{K}_{\lambda\alpha} \geq 1$. By Corollary 2.14, there exists some partition μ such that λ/μ is a horizontal strip, $|\mu| = |\overline{\alpha}|$, and $\mathcal{K}_{\mu\overline{\alpha}} \geq 1$. The last condition is equivalent to $\overline{\alpha} \trianglelefteq \mu$.

Then, since λ/μ is a horizontal strip, by Proposition 3.27, there exists a vector v such that $\lambda(v) = \mu$. We can identify the operator N_λ restricted to $\mathbb{C}^n/\langle v\rangle_\lambda$ with the matrix N_μ in Jordan form by choosing a Jordan basis. By induction, we assume there exists some $F'_\bullet \in \mathrm{Pet}_\mu$ such that $N_\mu F'_i \subset F'_i$ if and only if $i \in PS(\overline{\alpha})$. Suppose that, in $\mathbb{C}^n/\langle v\rangle_\lambda$, an ordered basis of vectors that determines $F'_\bullet$ is

$$\left(\bar{b}_1, \quad \bar{b}_2, \quad \ldots\ldots, \quad \bar{b}_{|\overline{\alpha}|-1}, \quad \bar{b}_{|\overline{\alpha}|}\right).$$

Consider the quotient map

$$\pi : \mathbb{C}^n \to \mathbb{C}^n/\langle v\rangle_\lambda,$$

and for every $i \in |\overline{\alpha}|$, let b_i be an vector in the preimage $\pi^{-1}(\overline{b_i})$ in $\mathbb{C}$. Then, the operator N_λ acts on these vectors b_i such that $N_\lambda b_i = N_\mu \overline{b}_i$. By the inductive assumption, $N_\lambda b_i = 0$ if and only if $i \in PS(\overline{\alpha})$. More explicitly,

$$\mathrm{span}\{N_\lambda b_1,\ N_\lambda b_2,\ \ldots,\ N_\lambda b_i\} \subset \mathrm{span}\{b_1,\ b_2,\ \ldots,\ b_i\} \text{ if and only if } i \in PS(\overline{\alpha}).$$

Now, we let $F_\bullet$ be the flag determined by the ordered basis.

$$\left(v,\quad N_\lambda v,\quad \ldots,\quad N_\lambda^{\alpha_1-2}v,\quad N_\lambda^{\alpha_1-1}v,\quad b_1,\quad b_2,\quad \ldots\ldots,\quad b_{|\overline{\alpha}|-1},\quad b_{|\overline{\alpha}|}\right).$$

We can check that $F_\bullet$ satisfies the desired properties: For every $i \in [\alpha_1 - 1]$, we have $N_\lambda F_i = \mathrm{span}\{N_\lambda v, \ldots, N_\lambda^i v\}$. Hence,

$$N_\lambda F_i \not\subset F_i = \mathrm{span}\{v,\ N_\lambda v,\ \ldots,\ N_\lambda^{i-1}v\}, \text{ while } N_\lambda F_i \subset F_{i+1} = \mathrm{span}\{v,\ N_\lambda v,\ \ldots,\ N_\lambda^{i}v\}.$$

Also, since $N_\lambda^{\alpha_1}v = 0$, we have

$$N_\lambda F_{\alpha_1} = \mathrm{span}\{N_\lambda v, \ldots, N_\lambda^{\alpha_1-1}v\} \subseteq \mathrm{span}\{v, N_\lambda v, \ldots, N_\lambda^{\alpha_1}v\} = F_{\alpha_1}.$$

Therefore, $N_\lambda F_i \subset F_i$ if and only if $i \in PS(\alpha)$, and the construction is complete. □

5. Poincaré polynomial of $\mathrm{Pet}_{\lambda,\alpha}$

In this section, given a partition λ and a composition α of the same size, we provide a recursive formula to compute the Poincaré polynomial of $\mathrm{Pet}_{\lambda,\alpha}$.

5.1. **A recursive formula.** We define the coefficient polynomial in this subsection, which we will use to provide the recursive formula for $\mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha}; q^{1/2})$. Recall that for every horizontal strip λ/μ, $I_{\lambda/\mu}$ is the set that contains all the column indices of the boxes in the skew Young diagram λ/μ. In other words, $I_{\lambda/\mu}$ is the set of all column indices j such that $\lambda_j^* - \mu_j^* = 1$.

Definition 5.1. *The **coefficient polynomial** $f_{\lambda/\mu}(q)$ for the horizontal strip λ/μ is*

$$f_{\lambda/\mu}(q) := \frac{\prod_{i\in I_{\lambda/\mu}} g_i(q)}{q-1}, \text{ where } g_i(q) := \begin{cases} q^{\lambda_i^*}, & \text{if } i+1 \in I_{\lambda/\mu} \\ q^{\lambda_i^*} - q^{\lambda_{i+1}^*}, & \text{otherwise.} \end{cases} \tag{5.1}$$

By the definition and Proposition 3.29 and Corollary 3.32, the following proposition is immediate.

Proposition 5.2. *Given partitions λ and μ such that λ/μ is a horizontal strip, we have*

$$f_{\lambda/\mu} = \frac{\#\{v \in \mathbb{F}_q^n \setminus \{0\} : \lambda(v) = \mu\}}{q-1}, \tag{5.2}$$

and $f_{\lambda/\mu}$ is a polynomial whose leading term is $q^{\sum_{i=1}^{\ell(\lambda)} i(\lambda_i - \mu_i) - 1}$.

For every composition $\alpha \neq 0$ and integer i such that $i \le \alpha_1$, we let $\alpha^{(i)}$ be the composition obtained by replacing the first entry α_1 with $\alpha_1 - i$. Our main result (Theorem 1.4) states that we have the recursive formula

$$\mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha}; q^{1/2}) = \sum_{i=1}^{\alpha_1} \sum_{\mu\in\Gamma(\lambda,|\lambda|-i)} f_{\lambda/\mu}(q) \cdot \mathrm{Poin}(\mathrm{Pet}_{\mu,\alpha^{(i)}}; q^{1/2}), \tag{5.3}$$

Here, we take the convention that

$$\mathrm{Poin}(\mathrm{Pet}_{(0),(0)}; q^{1/2}) = \mathrm{Poin}(\mathrm{Pet}_{(0)}; q^{1/2}) = 1.$$

Recall that we assume $(n)/0$ is a horizontal strip. Therefore, $\mathrm{Poin}(\mathrm{Pet}_{(0),(0)}; q^{1/2})$ may show up in the recursion when $\lambda = (n)$. (See Example 5.7.) Trivially, we have $\mathrm{Poin}(\mathrm{Pet}_{(1),(1)}; q^{1/2}) = 1$, so this formula provides us with a practical way to compute the Poincaré polynomials of $\mathrm{Pet}_{\lambda,\alpha}$. We provide the proof in the next subsection. Below is an example.

Example 5.3. *Let $\lambda = (3,1)$ and $\alpha = (2,2)$. When we take $i = 1$ and $\alpha^{(1)} = (1,2)$, the set $\Gamma(\lambda,1)$ is $\{(3),(2,1)\}$. The coefficient polynomials according to these horizontal strips are*

$$f_{(3,1)/(3)} = \frac{q^2-q}{q-1} = q \quad \text{and} \quad f_{(3,1)/(2,1)} = \frac{(q-1)}{q-1} = 1.$$

When we take $i = 2$, $\alpha^{(2)} = (2)$, the set $\Gamma(\lambda,1)$ is $\{(2),(1,1)\}$. The coefficient polynomials according to these horizontal strips are

$$f_{(3,1)/(2)} = \frac{(q^2-q)(q-1)}{q-1} = q^2-q \quad \text{and} \quad f_{(3,1)/(1,1)} = \frac{q(q-1)}{q-1} = q.$$

Therefore, by Theorem 1.4,

$$\begin{aligned}\mathrm{Poin}(\mathrm{Pet}_{(3,1),(2,2)}; q^{1/2}) = \;& q\cdot \mathrm{Poin}(\mathrm{Pet}_{(3),(1,2)}; q^{1/2}) && +1\cdot \mathrm{Poin}(\mathrm{Pet}_{(2,1),(1,2)}; q^{1/2}) \\ & +(q^2-q)\cdot \mathrm{Poin}(\mathrm{Pet}_{(2),(2)}; q^{1/2}) && +q\cdot \mathrm{Poin}(\mathrm{Pet}_{(1,1),(2)}; q^{1/2})\end{aligned}$$

Continuing this recursion, we have

$$\mathrm{Poin}(\mathrm{Pet}_{(3,1),(2,2)}; q^{1/2}) = q^3 + 3q^2 + 3q + 1$$

By the definition of the Poincaré polynomial, this computation shows $\mathrm{Pet}_{(3,1),(2,2)}$ has dimension 3 and contains one maximal dimensional component.

Remark 5.4. *Notice that if $\alpha = (|\lambda|)$, then $\mathrm{Pet}_{\lambda,|\lambda|}$ becomes Pet_λ and $\mathcal{P}(\lambda,(|\lambda|))$ becomes the polynomial $\mathrm{Poin}(\mathrm{Pet}_\lambda; q^{1/2})$. As a special case of (5.3), we have*

$$\mathrm{Poin}(\mathrm{Pet}_\lambda; q^{1/2}) = \sum_{\mu\in\Gamma(\lambda)} f_{\lambda/\mu}(q)\cdot \mathrm{Poin}(\mathrm{Pet}_\mu; q^{1/2}). \tag{5.4}$$

Example 5.5. *Let $\lambda = (3,2,1)$. The set $\Gamma\big((3,2,1)\big)$ is*

$$\big\{(3,1,1),(2,2,1),(2,1,1),(3,2),(3,1),(2,2),(2,1)\big\}.$$

The coefficient polynomials are

$$\begin{aligned} f_{(3,2,1)/(3,1,1)} &= \frac{q^2-1}{q-1} = q & f_{(3,2,1)/(2,2,1)} &= \frac{(q-1)}{q-1} = 1 \\ f_{(3,2,1)/(2,1,1)} &= \frac{q^2(q-1)}{q-1} = q^2 & f_{(3,2,1)/(3,2)} &= \frac{(q^3-q^2)}{q-1} = q^2 \\ f_{(3,2,1)/(3,1)} &= \frac{q^3(q^2-q)}{q-1} = q^4 & f_{(3,2,1)/(2,2)} &= \frac{(q^3-q^2)(q-1)}{q-1} = q^3-q^2 \\ f_{(3,2,1)/(2,1)} &= \frac{q^3q^2(q-1)}{q-1} = q^5. \end{aligned}$$

Therefore, by (5.4), we have

$$\begin{aligned}\mathrm{Poin}(\mathrm{Pet}_{(3,2,1)}; q^{1/2}) = \;& q \cdot \mathrm{Poin}(\mathrm{Pet}_{(3,1,1)}; q^{1/2}) && +1 \cdot \mathrm{Poin}(\mathrm{Pet}_{(2,2,1)}; q^{1/2}) \\ & +q^2 \cdot \mathrm{Poin}(\mathrm{Pet}_{(2,1,1)}; q^{1/2}) && +q^2 \cdot \mathrm{Poin}(\mathrm{Pet}_{(3,2)}; q^{1/2}) \\ & +q^4 \cdot \mathrm{Poin}(\mathrm{Pet}_{(3,1)}; q^{1/2}) && +(q^3 - q^2) \cdot \mathrm{Poin}(\mathrm{Pet}_{(2,2)}; q^{1/2}) \\ & +q^5 \cdot \mathrm{Poin}(\mathrm{Pet}_{(2,1)}; q^{1/2}).\end{aligned}$$

Continuing this recursion, we have

$$\mathrm{Poin}(\mathrm{Pet}_{(3,2,1)}; q^{1/2}) = 8q^7 + 31q^6 + 49q^5 + 45q^4 + 29q^3 + 14q^2 + 5q + 1.$$

By the definition of the Poincaré polynomial, this computation shows $\mathrm{Pet}_{(3,2,1)}$ *has dimension* 7 *and contains* 8 *maximal dimensional components.*

The next two corollaries show that we can recover the known results of the Peterson variety and the flag variety using Equation (5.4).

Corollary 5.6. *If* $\lambda = (n)$ *where* $n > 1$*, by Equation (5.4), we have*

$$\mathrm{Poin}(\mathrm{Pet}_{(n)}; q^{1/2}) = \sum_{i=0}^{n-1} q^{n-i-1} \,\mathrm{Poin}(\mathrm{Pet}_{(i)}; q^{1/2}).$$

Inductively, assume we have shown that $\mathrm{Poin}(\mathrm{Pet}_{(i)}; q^{1/2}) = (q+1)^{i-1}$ *for all* $1 < i < n$*. Then, we have*

$$\begin{aligned}\mathrm{Poin}(\mathrm{Pet}_{(n)}; q^{1/2}) &= \sum_{i=0}^{n-1} q^{n-i-1} \,\mathrm{Poin}(\mathrm{Pet}_{(i)}; q^{1/2}) \\ &= 1 + \sum_{i=1}^{n-1} q^{n-i-1} \,\mathrm{Poin}(\mathrm{Pet}_{(i)}; q^{1/2}) \\ &= 1 + \sum_{i=1}^{n-1} q^{n-i-1} \cdot (q+1)^{i-1} = (q+1)^{n-1}\end{aligned}$$

which is the known result on the Poincaré polynomial of the Peterson variety; see [13, Theorem 25].

Example 5.7. *We use Equation (5.4) to compute the first several terms of* $\mathrm{Poin}(\mathrm{Pet}_{(n)}; q^{1/2})$.

$$\begin{aligned}
&\mathrm{Poin}(\mathrm{Pet}_{(1)}; q^{1/2}) = \mathrm{Poin}(\mathrm{Pet}_{(0)}; q^{1/2}) = 1, \\
&\mathrm{Poin}(\mathrm{Pet}_{(2)}; q^{1/2}) = q^{2-1} \,\mathrm{Poin}(\mathrm{Pet}_{(0)}; q^{1/2}) + q^{2-2} \,\mathrm{Poin}(\mathrm{Pet}_{(1)}; q^{1/2}) = q + 1 \\
&\mathrm{Poin}(\mathrm{Pet}_{(3)}; q^{1/2}) = q^{3-1} \,\mathrm{Poin}(\mathrm{Pet}_{(0)}; q^{1/2}) + q^{3-2} \,\mathrm{Poin}(\mathrm{Pet}_{(1)}; q^{1/2}) + q^{3-3} \,\mathrm{Poin}(\mathrm{Pet}_{(2)}; q^{1/2}) \\
&\qquad = q^2 + q + q + 1 = q^2 + 2q + 1, \\
&\mathrm{Poin}(\mathrm{Pet}_{(4)}; q^{1/2}) = q^{4-1} \,\mathrm{Poin}(\mathrm{Pet}_{(0)}; q^{1/2}) + q^{4-2} \,\mathrm{Poin}(\mathrm{Pet}_{(1)}; q^{1/2}) + q^{4-3} \,\mathrm{Poin}(\mathrm{Pet}_{(2)}; q^{1/2}) \\
&\qquad +q^{4-4} \,\mathrm{Poin}(\mathrm{Pet}_{(3)}; q^{1/2}) = q^3 + 3q^2 + 3q + 1.
\end{aligned}$$

Corollary 5.8. *If* $\lambda = (1, 1, \ldots, 1)$ *with* n *entries, then note that* $\mathrm{Pet}_\lambda = \mathrm{Fl}(n)$*. By Theorem 1.4, we have*

$$\mathrm{Poin}(\mathrm{Fl}(n); q^{1/2}) = \frac{q^n - 1}{q - 1} \,\mathrm{Poin}(\mathrm{Fl}(n-1); q^{1/2})$$

Using induction on n*, we deduce that* $\mathrm{Poin}(\mathrm{Fl}(n); q^{1/2}) = \prod_{i=1}^{n} \frac{q^i - 1}{q - 1}$*, recovering the well-known formula for the Poincaré polynomial of the flag variety.*

5.2. **Proof of Theorem 1.4.** In this subsection, we provide a proof of Theorem 1.4. We use the counting method developed by Escobar, Precup, and Shareshian in [5]. As a special case of their result, we have the following.

Proposition 5.9 (Theorem 3.6 of [5])**.** *Given partition λ and composition α such that $|\lambda| = |\alpha|$, we have*

$$\operatorname{Poin}(\operatorname{Pet}_{\lambda,\alpha}; q^{1/2}) = \#\ \operatorname{Pet}_{\lambda,\alpha}(\mathbb{F}_q) \tag{5.5}$$

where $\operatorname{Pet}_{\lambda,\alpha}(\mathbb{F}_q)$ *is the set of all flags* $F_\bullet = (F_1 \subset \cdots \subset F_n = \mathbb{F}_q^n)$ *with* $\dim F_i = i$ *such that $N_\lambda F_i \subset F_i$ if $i \in PS(\alpha)$ and $N_\lambda F_i \subset F_{i+1}$ otherwise.*

In the rest of this subsection, we fix λ to be a partition and α to be a composition such that $|\lambda| = |\alpha| > 0$. For every nonzero vector $v \in \mathbb{F}_q^n$, we define $\mathcal{Z}_v$ as the set

$$\mathcal{Z}_v := \left\{F_\bullet \in \operatorname{Pet}_{\lambda,\alpha}(\mathbb{F}_q) : F_1 = \operatorname{span}\{v\}\right\}$$

For nonzero vectors v_1 and v_2 in $\mathbb{F}_q^n$ such that both $\mathcal{Z}_{v_1}$ and $\mathcal{Z}_{v_2}$ are nonempty, if $v_1 = cv_2$ for some nonzero $c \in \mathbb{F}_q^\times$, then $\mathcal{Z}_{v_1}$ and $\mathcal{Z}_{v_2}$ are equal sets; otherwise, $\mathcal{Z}_{v_1}$ and $\mathcal{Z}_{v_2}$ are disjoint. Therefore, every flag $F_\bullet \in \operatorname{Pet}_{\lambda,\alpha}(\mathbb{F}_q)$ appears exactly $q-1$ times in the disjoint union $\bigsqcup_{v\in\mathbb{F}_q^n} \mathcal{Z}_v$. Hence,

$$\#\operatorname{Pet}_{\lambda,\alpha}(\mathbb{F}_q) = \sum_{v\in\mathbb{F}_q^n\setminus\{0\}} \frac{\#\mathcal{Z}_v}{q-1}. \tag{5.6}$$

In the next lemma, we provide the criteria for $\mathcal{Z}_v$ to be nonempty and, when $\mathcal{Z}_v$ is nonempty. We also compute the cardinality of $\mathcal{Z}_v$. We take the convention that $\operatorname{Pet}_{(0),(0)}(\mathbb{F}_q)$ is the set containing the zero vector space. Hence, $\#\operatorname{Pet}_{(0),(0)}(\mathbb{F}_q) = 1$.

Lemma 5.10. *For every vector $v \in \mathbb{F}_q^n \setminus \{0\}$, the set $\mathcal{Z}_v$ is nonempty if and only if* $\dim\langle v\rangle_\lambda \leq \alpha_1$. *When $\mathcal{Z}_v$ is not empty, its cardinality equals* $\#\operatorname{Pet}_{\mu,\beta}(\mathbb{F}_q)$, *where $\mu = \lambda(v)$ and $\beta = \alpha^{(\dim\langle v\rangle_\lambda)}$.*

Proof. For every flag $F_\bullet \in \operatorname{Pet}_{\lambda,\alpha}(\mathbb{F}_q)$ such that $F_1 = \operatorname{span}\{v\}$, we first show that $\dim\langle v\rangle_\lambda \leq \alpha_1$. Suppose $\beta = \alpha_\lambda(F_\bullet)$. Then β has to be a refinement of α, and hence $\beta_1 \leq \alpha_1$. By Remark 4.13, we further have $\dim\langle v\rangle_\lambda = \beta_1 \leq \alpha_1$.

Now, suppose v is a nonzero vector such that $\dim\langle v\rangle_\lambda = \ell \leq \alpha_1$. We first deal with one special case: when $\lambda = \alpha = (n)$, and v is a vector in $\mathbb{F}_q^n$ such that $\dim\langle v\rangle_\lambda = n$. Then, we have $\lambda(v) = \alpha^{(n)} = (0)$. since $\alpha_\lambda(v) = (n)$, by Lemma 4.12 $\mathcal{Z}_v$ can only contain one flag determined by the ordered basis

$$\left(v,\ N_{(n)}v,\ \ldots,\ {N_{(n)}}^{n-1}v\right).$$

Therefore, in this case, $\mathcal{Z}_v$ is not empty, and its cardinality equals $\#\operatorname{Pet}_{(0),(0)}(\mathbb{F}_q)$, which is 1.

In other cases, notice that we must have $|\lambda(v)| = |\alpha^{(\ell)}| > 0$. Consider the map ψ

$$\begin{array}{rcl} \psi : \mathcal{Z}_v & \to & \operatorname{Pet}_{\lambda(v),\alpha^{(\ell)}}(\mathbb{F}_q) \\ F_\bullet & \mapsto & \left(F_{\ell+1}/F_\ell \subseteq \cdots \subseteq F_n/F_\ell\right). \end{array}$$

To check that this map is well-defined, notice that the linear transformation of $\mathbb{F}_q^n/F_\ell$ induced by N_λ is $N_{\lambda(v)}$. And for every i such that $i \leq n-\ell$, we have

$$N_{\lambda(v)}(F_{\ell+i}/F_\ell) = N_\lambda(F_{\ell+i}/F_\ell) \subseteq F_{\ell+i}/F_\ell,$$

if $\ell + i \in PS(\alpha)$; otherwise,

$$N_{\lambda(v)}(F_{\ell+i}/F_\ell) = N_\lambda(F_{\ell+i}/F_\ell) \subseteq F_{\ell+i+1}/F_\ell.$$

Therefore, $\big(F_{\ell+1}/F_\ell \subseteq \cdots \subseteq F_n/F_\ell\big)$ is a flag in $\mathrm{Pet}_{\lambda(v),\alpha^{(\ell)}}(\mathbb{F}_q)$.

The map ψ is clearly injective. Now, we show that this map is surjective by constructing an inverse map. For each flag $F'_\bullet$ in $\mathrm{Pet}_{\lambda(v),\alpha^{(\ell)}}$, we define a flag $F_\bullet \in \mathcal{Z}_v$ by letting

$$F_i = \mathrm{span}\{v, N_\lambda v, \ldots, N_\lambda^{i-1} v\} \qquad \text{for } i \in \{1, \ldots, \ell\};$$

And for $i \in \{\ell+1, \ldots, n\}$, F_i is the space such that $F_\ell \subseteq F_i$ and $F_i/F_\ell = F'_{i-\ell}$.

Now, we check that $F_\bullet \in \mathcal{Z}_v$. For every $i < \ell$, we have $N_\lambda F_i = \mathrm{span}\{N_\lambda v, \ldots, N_\lambda^i v\}$. Hence,

$$N_\lambda F_i \not\subset F_i = \mathrm{span}\{v,\ N_\lambda v,\ \ldots,\ N_\lambda^{i-1} v\},\ \text{while } N_\lambda F_i \subset F_{i+1} = \mathrm{span}\{v,\ N_\lambda v,\ \ldots,\ N_\lambda^{i} v\}.$$

Also, $N_\lambda F_\ell = \mathrm{span}\{N_\lambda v,\ \ldots,\ N_\lambda^{\ell-1} v\} \subseteq \mathrm{span}\{v,\ N_\lambda v,\ \ldots,\ N_\lambda^{\ell-1} v\} = F_\ell$.

For every i such that $i \le n - \ell$, we have

$$N_\lambda(F_{\ell+i}) = N_\lambda(F'_i + F_\ell) = N_\lambda(F'_i) + N_\lambda(F_\ell) = N_{\lambda(v)}(F'_i) + F_\ell \subseteq F'_i + F_\ell = F_{\ell+i}$$

if $\ell + i \in PS(\alpha)$; otherwise,

$$N_\lambda(F_{\ell+i}) = N_\lambda(F'_i + F_\ell) = N_\lambda(F'_i) + N_\lambda(F_\ell) = N_{\lambda(v)}(F'_i) + F_\ell \subseteq F'_{i+1} + F_\ell = F_{\ell+i+1}.$$

Therefore, we have constructed a flag $F_\bullet \in \mathrm{Pet}_{\lambda,\alpha}(\mathbb{F}_q)$ with $\psi(F_\bullet) = F'_\bullet$ and F_1 spanned by v. So, ψ has an inverse map and $\#\mathcal{Z}_v = \#\,\mathrm{Pet}_{\lambda(v),\alpha^{(\ell)}}(\mathbb{F}_q)$. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Proposition 5.9, to prove the statement on the Poincaré polynomial, it suffices to prove we have the counting result

$$\#\,\mathrm{Pet}_{\lambda,\alpha}(\mathbb{F}_q) = \sum_{i=1}^{\alpha_1} \sum_{\mu \in \Gamma(\lambda, |\lambda| - i)} f_{\lambda/\mu}(q) \cdot \#\,\mathrm{Pet}_{\mu,\alpha^{(i)}}(\mathbb{F}_q)$$

To show this, by Equation (5.6) and Lemma 5.10, we have

$$\#\,\mathrm{Pet}_{\lambda,\alpha}(\mathbb{F}_q) = \sum_{\substack{v \in \mathbb{F}_q^n: \\ \dim\langle v\rangle_\lambda \le \alpha_1}} \frac{\#\mathcal{Z}_v}{q-1} = \sum_{i=1}^{\alpha_1} \sum_{\substack{v \in \mathbb{F}_q^n: \\ \dim\langle v\rangle_\lambda = i}} \frac{\#\mathcal{Z}_v}{q-1}. \tag{5.7}$$

By Lemma 5.10, for each integer $i \le \alpha_1$ and every nonzero vector v such that $\dim\langle v\rangle_\lambda = i$, we have

$$\#\mathcal{Z}_v = \#\,\mathrm{Pet}_{\lambda(v),\alpha^{(i)}}(\mathbb{F}_q), \tag{5.8}$$

Now, substituting (5.8) into the right-hand-side of (5.7), we have

$$\#\,\mathrm{Pet}_{\lambda,\alpha}(\mathbb{F}_q) = \sum_{i=1}^{\alpha_1} \sum_{\substack{v \in \mathbb{F}_q^n: \\ \dim\langle v\rangle_\lambda = i}} \frac{\#\,\mathrm{Pet}_{\lambda(v),\alpha^{(i)}}(\mathbb{F}_q^n)}{q-1}. \tag{5.9}$$

While, by Proposition 3.27, for every partition μ, there exists some vector v such that $\lambda(v) = \mu$ if and only if λ/μ is a horizontal strip. Therefore, we can rewrite

(5.9) to be

$$\# \operatorname{Pet}_{\lambda,\alpha}(\mathbb{F}_q) = \sum_{i=1}^{\alpha_1} \sum_{\mu \in \Gamma(\lambda,|\lambda|-i)} \frac{\#\{v \in \mathbb{F}_q^n : \lambda(v) = \mu\}}{q-1} \cdot \# \operatorname{Pet}_{\mu,\alpha^{(i)}}(\mathbb{F}_q), \tag{5.10}$$

The desired formula now follows from Proposition 5.2. □

5.3. **Dimension formula and number of maximal dimensional components.** Recall Theorem 1.3 states that, given the partition λ and composition α such that $|\lambda| = |\alpha|$ and $\alpha \trianglelefteq \lambda$, The variety $\operatorname{Pet}_{\lambda,\alpha}$ has $\mathcal{K}_{\lambda\alpha}$ different maximum dimensional components, which are of the dimension

$$\dim(\operatorname{Pet}_{\lambda,\alpha}) = \sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\alpha) \tag{5.11}$$

Below is an example.

Example 5.11. *Let $\lambda = (4,2)$ and $\alpha = (2,1,3)$, note that $\alpha \trianglelefteq \lambda$. As checked in Example 2.12, $\mathcal{K}_{(4,2)(2,1,3)} = 2$. Hence, by Theorem 1.3, $\operatorname{Pet}_{(4,2)(2,1,3)}$ has dimension $4 + 2 \times 2 - 3 = 5$ and 2 maximal dimensional components.*

Remark 5.12. *When $\alpha = (1,1,\dots,1)$, $\operatorname{Pet}_{\lambda,\alpha}$ is the Springer fiber $\mathcal{B}_\lambda$. Then, this theorem recovers a weak version of the result that $\mathcal{B}_\lambda$ has dimension $\sum_{i=1}^{\ell(\lambda)}(i-1)\lambda_i$ and its components are in bijection with standard Young tableaux of shape λ [11].*

We will use induction to prove 1.3. Before that, we need one intermediate lemma.

Lemma 5.13. *Given any partition $\lambda \neq 0$, suppose for all compositions $\beta \in \mathcal{D}_\lambda$, we have $\dim \operatorname{Pet}_{\lambda,\beta} = \sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\beta)$. Then, for every composition α such that $|\alpha| = |\lambda|$ and α is not dominated by λ, we have*

$$\dim \operatorname{Pet}_{\lambda,\alpha} < \sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\alpha). \tag{5.12}$$

In other words, for all the compositions α such that $|\alpha| = |\lambda|$, we have $\dim \operatorname{Pet}_{\lambda,\alpha} \leq \sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\alpha)$, and the equality holds if and only if $\alpha \in \mathcal{D}_\lambda$.

Proof. Suppose α is a composition such that $|\alpha| = |\lambda|$ and α is not dominated by $\trianglelefteq\lambda$. By Theorem 1.2, we have the admissible decomposition

$$\operatorname{Pet}_{\lambda,\alpha} = \bigcup_{\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \operatorname{Pet}_{\lambda,\beta}.$$

Then, we have

$$\dim \operatorname{Pet}_{\lambda,\alpha} = \max_{\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \operatorname{Pet}_{\lambda,\beta}.$$

By assumption, for every $\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}$, we have $\dim \operatorname{Pet}_{\lambda,\beta} = \sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\beta)$. Substituting this into the above equation yields

$$\dim \operatorname{Pet}_{\lambda,\alpha} = \max_{\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \left(\sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\beta) \right). \tag{5.13}$$

Now, to obtain the desired inequality, it suffices to show that for all $\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}$, we have $\ell(\beta) > \ell(\alpha)$. Notice that, by assumption $\alpha \notin \mathcal{D}_\lambda$, every composition

$\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}$ is a refinement of α such that $\beta \neq \alpha$. Therefore, we must have $\ell(\beta) > \ell(\alpha)$ and $\dim \mathrm{Pet}_{\lambda,\alpha} < \sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\alpha)$. □

Now we are ready to prove Theorem 1.3 using Theorem 1.4. By definition, the leading degree of $\mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha}; q^{1/2})$ is equal to the dimension of $\mathrm{Pet}_{\lambda,\alpha}$, and the leading coefficient of $\mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha}; q^{1/2})$ equals the number of maximal dimensional components of $\mathrm{Pet}_{\lambda,\alpha}$. Hence, Theorem 1.4 provides a way to prove the claim about the dimension and maximal dimensional components of $\mathrm{Pet}_{\lambda,\alpha}$. Below, for a polynomial g, we let $\mathrm{LC}\ \ g$ be the leading coefficient of g.

Proof of Theorem 1.3. Let the notation be the same as the statement of Theorem 1.3. By the definition of $\mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha}; q^{1/2})$, it suffices to show that its degree is $\sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\alpha)$ and that its leading coefficient is $\mathcal{K}_{\lambda\alpha}$. Let $n = |\lambda|$. We finish the proof by induction on n.

Base case: When $n = 0$, we must have $\lambda = \alpha = 0$. By our convention, $\mathcal{P}((0),(0))$ is 1. Therefore, $\deg \mathcal{P}((0),(0))$ equals 0 and $\mathrm{LC}\ \ \mathcal{P}((0),(0))$ equals $\mathcal{K}_{(0)(0)} = 1$.

Inductive step: When $n > 0$, by Theorem 1.4 , we have

$$\begin{aligned}
&\deg \mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha}; q^{1/2}) \\
&= \deg \left(\sum_{i=1}^{\alpha_1} \sum_{\mu \in \Gamma(\lambda, |\lambda| - i)} f_{\lambda/\mu}(q) \cdot \mathrm{Poin}(\mathrm{Pet}_{\mu,\alpha^{(i)}}; q^{1/2}) \right) \\
&= \max_{i \in [\alpha_1]} \max_{\mu \in \Gamma(\lambda, |\lambda| - i)} \left(\deg f_{\lambda/\mu}(q) \cdot \mathrm{Poin}(\mathrm{Pet}_{\mu,\alpha^{(i)}}; q^{1/2}) \right) \\
&= \max_{i \in [\alpha_1]} \max_{\mu \in \Gamma(\lambda, |\lambda| - i)} \left(\deg f_{\lambda/\mu}(q) + \deg \mathrm{Poin}(\mathrm{Pet}_{\mu,\alpha^{(i)}}; q^{1/2}) \right).
\end{aligned} \tag{5.14}$$

For every integer $i \in [\alpha_1]$ and $\mu \in \Gamma(\lambda, |\lambda| - i)$, by Proposition 5.2,

$$\deg f_{\lambda/\mu}(q) = \sum_{j=1}^{\ell(\lambda)} j(\lambda_j - \mu_j) - 1. \tag{5.15}$$

Meanwhile, by the inductive assumption and Lemma 5.13, we have

$$\deg \mathrm{Poin}(\mathrm{Pet}_{\mu,\alpha^{(i)}}; q^{1/2}) \leq \sum_{j=1}^{\ell(\mu)} j\mu_j - \ell(\alpha^{(i)}) = \sum_{i=j}^{\ell(\lambda)} i\mu_j - \ell(\alpha^{(i)}), \tag{5.16}$$

where the equality holds if and only if $\alpha^{(i)} \in \mathcal{D}_\mu$. Therefore, combining Equations (5.14), (5.15) and 5.16, we

$$\begin{aligned}
\deg \operatorname{Poin}(\operatorname{Pet}_{\lambda,\alpha}; q^{1/2}) &= \max_{i\in[\alpha_1]} \max_{\mu\in\Gamma(\lambda,|\lambda|-i)} \left(\deg f_{\lambda/\mu}(q) + \deg \operatorname{Poin}(\operatorname{Pet}_{\mu,\alpha^{(i)}}; q^{1/2})\right) \\
&\leq \max_{i\in[\alpha_1]} \max_{\mu\in\Gamma(\lambda,|\lambda|-i)} \left(\sum_{j=1}^{\ell(\lambda)} j(\lambda_j-\mu_j) + \sum_{j=1}^{\ell(\lambda)} j\mu_j - \ell(\alpha^{(i)}) - 1\right) \\
&= \max_{i\in[\alpha_1]} \max_{\mu\in\Gamma(\lambda,|\lambda|-i)} \left(\sum_{j=1}^{\ell(\lambda)} j\lambda_j - \ell(\alpha^{(i)}) - 1\right) \\
&= \max_{i\in[\alpha_1]} \left(\sum_{j=1}^{\ell(\lambda)} j\lambda_j - \ell(\alpha^{(i)}) - 1\right).
\end{aligned}$$

Note that, $\ell(\alpha^{(i)}) = \ell(\alpha)$ if $i \neq \alpha_1$, while $\ell(\alpha^{(i)}) = \ell(\alpha) - 1$ if $i = \alpha_1$. Therefore, we have

$$\deg \operatorname{Poin}(\operatorname{Pet}_{\lambda,\alpha}; q^{1/2}) \leq \sum_{j=1}^{\ell(\lambda)} j\lambda_j - \ell(\alpha). \tag{5.17}$$

On the other hand, note that when $i = \alpha_1$, $\alpha^{(i)}$ is $\overline{\alpha}$ which is the composition obtained by deleting the first entry from α. Since $\alpha \trianglelefteq \lambda$, by Corollary 2.14, there exists some partition $\nu \in \Gamma(\lambda, |\overline{\alpha}|)$ such that $\overline{\alpha} \trianglelefteq \nu$. Since $|\overline{\alpha}| < |\alpha| = n$, inductively we can assume

$$\deg \mathcal{P}(\nu, \overline{\alpha}) = \sum_{j=1}^{\ell(\nu)} j\nu_j - \ell(\overline{\alpha}) = \sum_{j=1}^{\ell(\lambda)} j\nu_j - \ell(\overline{\alpha}).$$

Therefore,

$$\begin{aligned}
&\deg \operatorname{Poin}(\operatorname{Pet}_{\lambda,\alpha}; q^{1/2}) \\
&= \max_{i\in[\alpha_1]} \max_{\mu\in\Gamma(\lambda,|\lambda|-i)} \left(\deg f_{\lambda/\mu}(q) + \deg \operatorname{Poin}(\operatorname{Pet}_{\mu,\alpha^{(i)}}; q^{1/2})\right) \\
&\geq \deg f_{\lambda/\nu}(q) + \deg \mathcal{P}(\nu, \overline{\alpha}) \\
&= \sum_{j=1}^{\ell(\lambda)} j(\lambda_j - \nu_j) + \sum_{j=1}^{\ell(\lambda)} j\nu_j - \ell(\overline{\alpha}) - 1 \\
&= \sum_{j=1}^{\ell(\lambda)} j\lambda_j - \ell(\overline{\alpha}) - 1 \\
&= \sum_{j=1}^{\ell(\lambda)} j\lambda_j - \ell(\alpha).
\end{aligned} \tag{5.18}$$

Combining Formulas (5.17) and (5.18), we conclude that $\deg \operatorname{Poin}(\operatorname{Pet}_{\lambda,\alpha}; q^{1/2}) = \sum_{j=1}^{\ell(\lambda)} j\lambda_j - \ell(\alpha)$.

Now we want to show the leading coefficient of $\mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha};q^{1/2})$ is $\mathcal{K}_{\lambda\alpha}$. By Proposition 5.2, for every $\mu\in\Gamma(\lambda)$, LC $f_{\lambda/\mu}(q)=1$. Since

$$\deg \mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha};q^{1/2}) = \deg\left(\sum_{i=1}^{\alpha_1}\sum_{\mu\in\Gamma(\lambda,|\lambda|-i)} f_{\lambda/\mu}(q)\cdot \mathrm{Poin}(\mathrm{Pet}_{\mu,\alpha^{(i)}};q^{1/2})\right) = \sum_{j=1}^{\ell(\lambda)} j\lambda_j - \ell(\alpha),$$

while for every $i\in[\alpha_1]$ and $\mu\in\Gamma(\lambda,|\lambda|-i)$, the equation

$$\deg\Big(f_{\lambda/\mu}(q)\cdot \mathrm{Poin}(\mathrm{Pet}_{\mu,\alpha^{(i)}};q^{1/2})\Big) = \sum_{j=1}^{\ell(\lambda)} j\lambda_j - \ell(\alpha)$$

holds if and only if $\alpha^{(i)}=\overline{\alpha}$, $\mu\in\Gamma(\lambda,|\overline{\lambda}|)$, and $\overline{\alpha}\trianglelefteq\mu$. Therefore,

$$\begin{aligned}\text{LC }\ \mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha};q^{1/2}) &= \sum_{\substack{\mu\in\Gamma(\lambda,|\overline{\lambda}|),\\ \overline{\alpha}\trianglelefteq\mu}} \text{LC }\ \Big(f_{\lambda/\mu}(q)\cdot \mathrm{Poin}(\mathrm{Pet}_{\mu,\overline{\alpha}};q^{1/2})\Big)\\ &= \sum_{\substack{\mu\in\Gamma(\lambda,|\overline{\lambda}|),\\ \overline{\alpha}\trianglelefteq\mu}} \text{LC }\ \mathrm{Poin}(\mathrm{Pet}_{\mu,\overline{\alpha}};q^{1/2}).\end{aligned}$$

Inductively, assume that for every $\mu\in\Gamma(\lambda,|\overline{\lambda}|)$ and $\overline{\alpha}\trianglelefteq\mu$, we have LC $\mathcal{P}(\mu,\overline{\alpha})=\mathcal{K}_{\mu\overline{\alpha}}$. Then, by Equation (2.4), we have

$$\text{LC }\ \mathrm{Poin}(\mathrm{Pet}_{\lambda,\alpha};q^{1/2}) = \sum_{\substack{\mu\in\Gamma(\lambda,|\overline{\lambda}|),\\ \overline{\alpha}\trianglelefteq\mu}} \mathcal{K}_{\mu\overline{\alpha}} = \mathcal{K}_{\lambda\alpha}$$

which is the desired equation. ◻

We can obtain a more general statement on the dimension and number of maximal dimensional components of $\mathrm{Pet}_{\lambda,\alpha}$.

Theorem 5.14. *Given partition λ and composition α such that $|\lambda|=|\alpha|$, The dimension of $\mathrm{Pet}_{\lambda,\alpha}$ is given by the formula*

$$\dim \mathrm{Pet}_{\lambda,\alpha} = \sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\beta), \tag{5.19}$$

where β is an element in $(\mathcal{D}_\lambda\cap\mathcal{R}_\alpha)^{\min}$. The number of maximal dimensional components of $\mathrm{Pet}_{\lambda,\alpha}$ equals

$$\sum_{\beta\in(\mathcal{D}_\lambda\cap\mathcal{R}_\alpha)^{\min}} \mathcal{K}_{\lambda\beta}. \tag{5.20}$$

Since all elements in $(\mathcal{D}_\lambda\cap\mathcal{R}_\alpha)^{\min}$ have the same length, the Formula (5.19) is well-defined.

Proof. For every partition λ and composition α such that $|\lambda|=|\alpha|$, if $\alpha\trianglelefteq\lambda$, then $(\mathcal{D}_\lambda\cap\mathcal{R}_\alpha)^{\min}=\{\alpha\}$. The statement holds by Theorem 1.3.

In the rest of the proof, suppose α is not dominated by λ. Then, by Theorem 1.2, we have the admissible decomposition

$$\mathrm{Pet}_{\lambda,\alpha} = \bigcup_{(\mathcal{D}_\lambda\cap\mathcal{R}_\alpha)^{\mathrm{crs}}} \mathrm{Pet}_{\lambda,\beta}$$

such that none of these varieties is contained within the others. Combining this with Theorem 1.3, we have

$$\dim \operatorname{Pet}_{\lambda,\alpha} = \max_{(\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \dim \operatorname{Pet}_{\lambda,\beta} = \max_{(\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}} \left(\sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\beta) \right).$$

Clearly, the right-hand side of the equation realizes the maximal value if and only if β is of minimal length in $(\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{crs}}$, which implies $\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{min}}$.

To prove the statement regarding the number of maximal dimensional components, note that by the previous part of the proof, all maximal dimensional components are inside the union

$$\bigcup_{(\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{min}}} \operatorname{Pet}_{\lambda,\beta} .$$

By Theorem 1.3 again, for every element $\beta \in (\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{min}}$, $\operatorname{Pet}_{\lambda,\beta}$ has $\mathcal{K}_{\lambda\beta}$ different maximal dimensional components, whose dimension also equals $\dim \operatorname{Pet}_{\lambda,\alpha}$. We want to show that if β and γ are different elements in $(\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{min}}$, then none of the maximal dimensional components of $\operatorname{Pet}_{\lambda,\alpha}$ is contained in their intersection. By Proposition 4.3,

$$\operatorname{Pet}_{\lambda,\beta} \cap \operatorname{Pet}_{\lambda,\gamma} = \operatorname{Pet}_{\lambda,\beta\circ\gamma} .$$

Note that $\beta \circ \gamma$ is in $(\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)$, and $\ell(\beta \circ \gamma) > \ell(\beta)$ since both β and γ are of minimal length, and $\beta \circ \gamma$ refines both of them. By Theorem 1.3 again, we have

$$\dim \operatorname{Pet}_{\lambda,\beta} \cap \operatorname{Pet}_{\lambda,\gamma} = \dim \operatorname{Pet}_{\lambda,\beta\circ\gamma} < \dim \operatorname{Pet}_{\lambda,\beta} = \dim \operatorname{Pet}_{\lambda,\alpha} .$$

Therefore, the number of maximal dimensional components of $\operatorname{Pet}_{\lambda,\alpha}$ is the same as the sum of the numbers of maximal dimensional components of every $\operatorname{Pet}_{\lambda,\beta}$, which is $\mathcal{K}_{\lambda\beta}$, where β runs over all elements in $(\mathcal{D}_\lambda \cap \mathcal{R}_\alpha)^{\mathrm{min}}$. □

Example 5.15. *Given $\lambda = (3,2)$ and $\alpha = (4,1)$, the set $\left(\mathcal{D}_{(3,2)} \cap \mathcal{R}_{4,1}\right)^{\mathrm{min}}$ contains $(3,1,1), (2,2,1)$ and $(1,3,1)$. Then, by Theorem 5.14, $\operatorname{Pet}_{(3,2),(4,1)}$ is of dimension $3 + 2 \times 2 - 3 = 4$. And since*

$$\begin{aligned} &\mathcal{K}_{(3,2)(3,1,1)} + \mathcal{K}_{(3,2)(2,2,1)} + \mathcal{K}_{(3,2)(1,3,1)} \\ &= \#\left\{ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \right\} + \#\left\{ \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} \right\} + \#\left\{ \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \right\} \\ &= 1 + 2 + 1 = 4, \end{aligned}$$

the variety $\operatorname{Pet}_{(3,2),(4,1)}$ has four different maximal dimensional components.

Recall that when ℓ is an integer, we use $\mathrm{SSYT}(\lambda, \ell)$ to denote the set of all semistandard Young tableaux whose content α satisfies $\ell(\alpha) = \ell$. Our Proposition 1.1 states that for every partition λ, the generalized Peterson variety $\operatorname{Pet}_\lambda$ has dimension

$$\dim (\operatorname{Pet}_\lambda) = \sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\lambda),$$

and its components of this maximum dimension are in bijection with the set $\mathrm{SSYT}(\lambda, \ell(\lambda))$. We show that Proposition 1.1 follows from Theorem 5.14.

Proof of Proposition 1.1. For every partition λ, notice that $\mathrm{Pet}_\lambda = \mathrm{Pet}_{\lambda,(n)}$ and

$$\big(\mathcal{D}_\lambda \cap \mathcal{R}_{(n)}\big)^{\min} = (\mathcal{D}_\lambda)^{\min} = \{\beta \in \mathcal{D}_\lambda : \ell(\beta) = \ell(\lambda)\}.$$

Therefore, by Theorem 5.14, we immediately have

$$\dim \mathrm{Pet}_\lambda = \dim \mathrm{Pet}_{\lambda,(n)} = \sum_{i=1}^{\ell(\lambda)} i\lambda_i - \ell(\lambda).$$

And the number of maximal dimensional components equals

$$\sum_{\beta\in\left(\mathcal{D}_\lambda\cap\mathcal{R}_{(n)}\right)^{\min}} \mathcal{K}_{\lambda\beta} = \sum_{\substack{\beta\in\mathcal{D}_\lambda:\\ \ell(\beta)=\ell(\lambda)}} \mathcal{K}_{\lambda\beta}.$$

Now, the desired formula follows from Equation (2.2). $\square$

Example 5.16. *When* $\lambda = (3,2,1)$*, Proposition 1.1 implies* $\dim \mathrm{Pet}_{(3,2,1)} = 3 + 2\times 2 + 1\times 3 - 3 = 7$*, and the number of maximal dimensional components equals* $\#\,\mathrm{SSYT}((3,2,1),3)$ *which is* 8 *as showed in Subsection 2.4. Notice this agrees with the computational result in Example 5.5.*

Appendix: Summary of notation

Notation	**Meaning**
α,β,γ	Integer compositions.
$\ell(\alpha)$	The length of α.
$\|\alpha\|$	The size of α.
$PS(\alpha)$	The partial sum set of composition α.
$\overline{\alpha}$	The composition obtained by deleting the first entry from α.
$\alpha^{(i)}$	The integer composition obtained by replacing the first entry α_1 of α with $\alpha_1 - i$.
$\beta \preceq \alpha$	The composition β refines the composition α.
$\mathcal{R}_\alpha$	The set of all compositions that refine α.
λ,μ,ν	Integer partitions.
λ^*	The transpose partition of the partition λ.
$\alpha \trianglelefteq \lambda$	The composition α is dominated by partition λ.
$\mathcal{D}_\lambda$	The set of all compositions dominated by λ.
A^{crs}	The subset of all the coarsest elements in set A where A is a set of compositions of the same size.
$A^{\min}$	The subset of all the minimal length elements in set A where A is a set of compositions of the same size.
λ/μ	The skew Young tableau defined by λ and μ.
$I_{\lambda/\mu}$	Given that λ/μ is a horizontal strip, $I_{\lambda/\mu}$ is the set of all the column indices of the boxes in the skew Young diagram λ/μ.

$\Gamma(\lambda)$	The set of all partitions μ such that λ/μ is a horizontal strip.
$\Gamma(\lambda, m)$	The set of all partitions μ such that λ/μ is a horizontal strip and $\lvert\mu\rvert = m$.
$\mathrm{SSYT}(\lambda, \alpha)$	The set of all semistandard Young tableaux of shape λ and content α.
$\mathcal{K}_{\lambda\alpha}$	The Kostka number, which is the cardinality of $\mathrm{SSYT}(\lambda, \alpha)$.
$\mathrm{SSYT}(\lambda, \ell)$	Given an integer ℓ, $\mathrm{SSYT}(\lambda, \ell)$ is the set of all semistandard Young tableaux whose content α satisfies $\ell(\alpha) = \ell$.
$\mathrm{Hess}(X, h)$	The Hessenberg variety defined by matrix X and Hessenberg function h.
$\mathcal{B}_\lambda$	The Springer fiber.
Pet_λ	The generalized Peterson variety.
$\mathrm{Pet}_{\lambda,\alpha}$	The generalized parabolic Peterson variety.
$\mathrm{Poin}(Z; t)$	The Poincaré polynomial of a variety Z.
$f_{\lambda/\mu}(q)$	The coefficient polynomial as defined in Subsection 5.1.
$\mathrm{ht}_\lambda(v)$	The height of a vector as defined in Subsection 3.1.
$\alpha_\lambda(F_\bullet)$	The composition associated to a flag $F_\bullet$ as defined in Subsection 4.1.

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