On Generalized Peterson Variety

AMS Special Session

Zijing Carole Zhuang

Department of Mathematics
Washington University in St. Louis

October 18, 2025

Notations and Definitions

Notation*:

- A **flag** $F_\bullet$ is a chain of vector subspaces such that $F_\bullet = (F_1 \subset \ldots \subset F_n = \mathbb{C}^n)$ with $\dim F_i = i$.
- $\mathrm{Fl}(n)$ is the **flag variety** of all flags.

Below for a fixed integer partition λ, let X be a **nilpotent matrix** of **Jordan type** λ.

Definition (Springer Fiber)

$\mathcal{B}_\lambda := \{F_\bullet \in \mathrm{Fl}(n) : XF_i \subseteq F_i\}$.

Definition (Generalized Peterson Variety)**

$\mathcal{P}\mathrm{et}_\lambda := \{F_\bullet \in \mathrm{Fl}(n) : XF_i \subseteq F_{i+1}\}$.

*: Everything in this talk is assumed to be in type A and field $\mathbb{C}$.

**: The usual definition of the Peterson variety requires X to be regular.

$\mathcal{B}_\lambda$: An Interplay of Geometry, Rep Theory, And Combinatorics

- [Springer, 1976]: $H^*(\mathcal{B}_\lambda)$ carries a S_n action.
- [Springer, 1976]: $H^{\text{top}}(\mathcal{B}_\lambda)$ is the irreducible representation S^λ of S_n.
- [Spaltenstein, 1976]: $\mathcal{B}_\lambda$ **pure dimensional** such that $\dim \mathcal{B}_\lambda = \sum(i-1)\lambda_i$;
- [Spaltenstein, 1976]: $\mathcal{B}_\lambda$ admits an **affine paving**. Components bijective with the set of **standard** Young tableaux.
- [Shimomura, 1980] When X chosen carefully, $\mathcal{B}_\lambda \cap C_w$ gives the affine paving, which are bijective with the set of **row-strict tableaux**.

E.g. $\mathcal{B}_{(2,1)}$ can be paved by one $\mathbb{C}^0$ and two $\mathbb{C}^1$ corresponding to tableaux

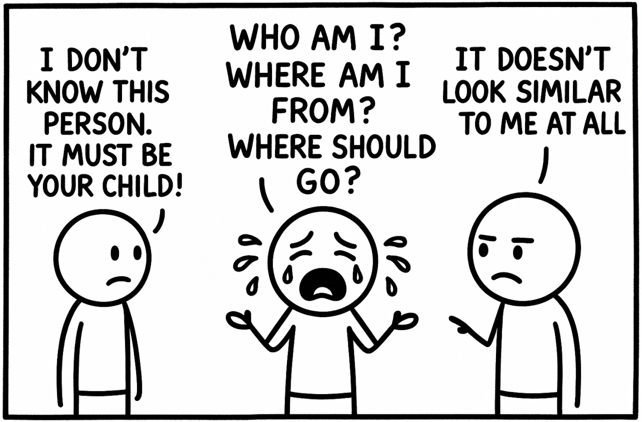
$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$\mathcal{P}\mathrm{et}_{(n)}$: Some Known Results

Introduced by Peterson in the 1990s, studied by [Kostant, 1996], [Insko and Yong, 2012] and others.

- $\mathcal{P}\mathrm{et}_{(n)}$ admits affine pavings.
- Irreducible with $\mathrm{Poin}(\mathcal{P}\mathrm{et}_n; q) = (1+q)^{n-1}$.
- Related to the quantum cohomology of **partial flag varieties**.
- Ring isomorphism $H^*(\mathcal{P}\mathrm{et}_{(n)}; \mathbb{C}) \cong H^*(\mathrm{Perm}_n; \mathbb{C})^{S_n}$

Pet_λ: "Who Am I?"



$\mathcal{B}_\lambda$ $\mathcal{P}\mathrm{et}_\lambda$ $\mathcal{P}\mathrm{et}_{(n)}$

Peterson-Tymoczko Tableau Correspondence

Theorem (Special case of [Tymoczko, 2006])

When X chosen carefully, $\mathcal{P}\mathrm{et}_\lambda$ admits an **affine paving**. *Each affine cell corresponds to the* **Young tableau** *of shape λ with filling $[n]$ such that if $\boxed{i \,|\, j}$, then either $i < j$ or $j = i - 1$. And the dimension of each cell equals the* **inversion number** *of the corresponding tableau.*

We may call these tableaux the **Peterson-Tymoczko tableaux**.

Examples (When $\lambda = (6, 3)$)

The maximal inversion number is 10, there are 4 Peterson-Tymoczko tableaux realizing this inversion number

$$\begin{array}{|c|c|c|c|c|c|}\hline 6&5&4&3&2&1\\\hline 9&8&7\\\cline{1-3}\end{array}\quad\begin{array}{|c|c|c|c|c|c|}\hline 5&4&3&2&1&6\\\hline 9&8&7\\\cline{1-3}\end{array}\quad\begin{array}{|c|c|c|c|c|c|}\hline 4&3&2&1&6&5\\\hline 9&8&7\\\cline{1-3}\end{array}\quad\begin{array}{|c|c|c|c|c|c|}\hline 3&2&1&6&5&4\\\hline 9&8&7\\\cline{1-3}\end{array}$$

Examples And Corollaries

Examples (When $\lambda = (3, 2, 1)$)

The maximal inversion number is 7, there are 8 Peterson-Tymoczko tableaux realize this inversion number:

$$
\begin{array}{|c|c|c|}\hline 3&2&1\\\hline 5&4\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 3&2&1\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 2&1&5\\\hline 4&3\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 2&1&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 2&1&3\\\hline 5&4\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 2&1&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 1&3&2\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array}
$$

Corollary given by affine paving

Let $\mathcal{T}$ run over all Peterson-Tymoczko tableaux of shape λ, then

$$\mathrm{Poin}(\mathcal{P}\mathrm{et}_\lambda; q) = \sum q^{\mathrm{inv}(\mathcal{T})}$$

$$\dim \mathcal{P}\mathrm{et}_\lambda = \max \mathrm{inv}(\mathcal{T})$$

$$\#\{\text{Maximal dimensional components of } \mathcal{P}\mathrm{et}_\lambda\} = \#\{\mathcal{T} \text{ of maximal inversion number}\}$$

Conjectures On Dimensions And $\#\{$Max Dim Components$\}$

Interpolating on data like these, I conjured that:

Theorem (Zhuang, 2025 expected)

Let k be the length of λ,

$$\dim \mathcal{P}\mathrm{et}_\lambda = \sum_{i=1}^{k}(\lambda_i - 1) = (\lambda_1 + 2\lambda_2 + \ldots k\lambda_k) - k.$$

Theorem (Zhuang, 2025 expected)

$\#\{$*Maximal dimensional components of* $\mathcal{P}\mathrm{et}_\lambda\} = \prod_{i<j\in[k]} \frac{\lambda_i - \lambda_j + j - i}{j-i}$.

How to interpret those formulas?

“Curiouser And Curiouser”

Theorem (Equivalent to the previous one)

Below are several numbers all equal to each other

- **Leading coefficient** *of* $\mathrm{Poin}(\mathcal{P}\mathrm{et}_\lambda; q) = \prod_{i<j\in[k]} \frac{\lambda_i-\lambda_j+j-i}{j-i}$.
- $\#\{$ **Peterson-Tymoczko tableaux** *of shape* λ *of* **maximal inversion number** $\}$.
- $\#\{$ **Maximal dimensional components** *of* $\mathcal{P}\mathrm{et}_\lambda\}$
- $\#\{$ **Semistandard Young tableaux*** *of shape* $\lambda\}$
- $\#\{$ *Basis of* **irreducible representation of** sl_r *characterized by* $\lambda\}$.

*: Column strictly increasing and row nondecreasing tableaux with filling $[k]$.

Examples (Semistandard Young tableaux of shape $\lambda = (3,2,1)$)

$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&2\\\cline{1-2} 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 2&3\\\cline{1-2} 3\\\cline{1-1}\end{array}$

sl_r Action on $H^*(\mathcal{P}\mathrm{et}_\lambda)$?

Open conjecture

- $H^*(\mathcal{P}\mathrm{et}_\lambda)$ carries an sl_r action.
- $H^{\mathsf{top}}(\mathcal{P}\mathrm{et}_\lambda)$ is the irreducible representation Γ_λ of sl_r.

The most similar result may be found in chapter 4 of [Chriss and Ginzburg, 1997] where they realize Γ_λ as the homology class of some subvariety of the **partial flag variety**.

Approach to a $\mathcal{B}_\lambda$-like theory of $\mathcal{P}\mathrm{et}_\lambda$

Inspired by [Spaltenstein, 1976] on $\mathcal{B}_\lambda$, I considered the projection

$$\pi : \mathcal{P}\mathrm{et}_\lambda \longrightarrow \mathbb{P}^{n-1}$$
$$(F_1 \subset \ldots \subset F_n) \mapsto \mathbb{P}(F_1)$$

Key Observation: The fiber $\pi^{-1}(v) \cong \mathcal{P}\mathrm{et}_\mu$ where μ is a smaller partition.

Examples

When $\lambda = (3,1)$, let e_1, e_2, e_3, e_4 be the Jordan basis of X. If $v = e_2 + e_4$, the Peterson condition forces every $(F_1 \subset F_2 \subset F_3 \subset F_4) \in \pi^{-1}(e_2 + e_4)$ satisfying

$$F_1 = \mathbb{C}\{e_2 + e_4\}, F_2 = \mathbb{C}\{e_2 + e_4, e_1\}$$

$$\overline{X}(F_3/F_2) \subset (F_4/F_2) : \overline{X} = X|_{\mathbb{C}^4/F_2} \text{ of Jordan type } (2)$$

It motivates us to decompose $\mathbb{P}^{n-1}$ into a disjoint union U_i such that

$$\pi^{-1}(U_i) \cong U_i \times \mathcal{P}\mathrm{et}_\mu$$

A Recursive Formula On $\mathrm{Poin}(\mathcal{P}\mathrm{et}_\lambda; q)$

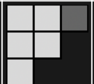
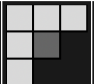
Theorem (Zhuang, 2025 expected)

Using the counting method in [Escobar et al., 2022],

$$\mathrm{Poin}(\mathcal{P}\mathrm{et}_\lambda; q) = \begin{cases} 1, & \textit{when } \lambda = (1) \\ \sum_\mu f_{\lambda/\mu}(q) \cdot \mathrm{Poin}(\mathcal{P}\mathrm{et}_\mu; q), & \textit{when } \lambda \neq (1) \end{cases}$$

Where μ *shows up in the recursion if* λ/μ *is a* **horizontal strip**. $f_{\lambda/\mu}(q)$ *is a polynomial of* q *whose leading term is* $q^{\sum_{i=1}^k i(\lambda_i - \mu_i) - 1}$.

Examples ($\lambda = (3, 2, 1)$)

{2, 1}	q^5		{2, 2}	$(-1 + q)\ q^2$		{3, 1}	q^4		{3, 2}	q^2		{2, 1, 1}	q^2	
{2, 2, 1}	1		{3, 1, 1}	q										

Examples Continued

In last example, we have

$$\mathrm{Poin}(\mathcal{P}\mathrm{et}_{(3,2,1)}; q) = 8q^7 + 31q^6 + 49q^5 + 45q^4 + 29q^3 + 14q^2 + 5q + 1$$

Compare the leading term with the result of Peterson-Tymoczko tableaux:

Examples (When $\lambda = (3, 2, 1)$)

The maximal inversion number is 7, there are 8 Peterson-Tymoczko tableaux realize this inversion number:

$$\begin{ytableau} 3 & 2 & 1 \\ 5 & 4 \\ 6 \end{ytableau} \quad \begin{ytableau} 3 & 2 & 1 \\ 4 & 5 \\ 6 \end{ytableau} \quad \begin{ytableau} 2 & 1 & 5 \\ 4 & 3 \\ 6 \end{ytableau} \quad \begin{ytableau} 2 & 1 & 4 \\ 3 & 5 \\ 6 \end{ytableau} \quad \begin{ytableau} 2 & 1 & 3 \\ 5 & 4 \\ 6 \end{ytableau} \quad \begin{ytableau} 2 & 1 & 3 \\ 4 & 5 \\ 6 \end{ytableau} \quad \begin{ytableau} 1 & 3 & 2 \\ 4 & 5 \\ 6 \end{ytableau} \quad \begin{ytableau} 1 & 2 & 4 \\ 3 & 5 \\ 6 \end{ytableau}$$

Apply The Recursive Formula To Special Cases

Regular case: when $\lambda = (n)$

By recursive formula, $\mathrm{Poin}(\mathcal{P}\mathrm{et}_{(n)}; q) = \sum_{i=1}^{n-1} q^{n-i-1} \mathrm{Poin}(\mathcal{P}\mathrm{et}_{(i)}; q)$.
Using induction, we have

$$\mathrm{Poin}(\mathcal{P}\mathrm{et}_{(n)}; q) = (q+1)^{n-1}$$

$FL(n)$ case: when $\lambda = (1, 1, \ldots, 1) = (1^n)$

Notice $\mathcal{P}\mathrm{et}_{(1^n)} = \mathrm{Fl}(n)$, by recursive formula:

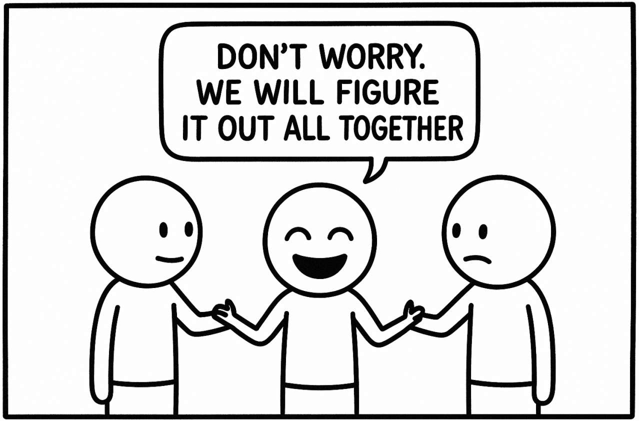
$$\mathrm{Poin}(\mathrm{Fl}(n); q) = \frac{q^n - 1}{q - 1} \mathrm{Poin}(\mathrm{Fl}(n-1); q)$$

Using induction, we have

$$\mathrm{Poin}(\mathrm{Fl}(n); q) = \prod_{i=1}^{n} \frac{q^i - 1}{q - 1}$$

Thank You!

The End Of The Talk; Not the End of The Story...



$\mathcal{B}_\lambda$ $\mathcal{P}\mathrm{et}_\lambda$ $\mathcal{P}\mathrm{et}_{(n)}$

References

Chriss, N. and Ginzburg, V. (1997).
Representation theory and complex geometry, volume 42.
Springer.

Escobar, L., Precup, M., and Shareshian, J. (2022).
Hessenberg varieties of codimension one in the flag variety.
arXiv preprint arXiv:2208.06299.

Insko, E. and Yong, A. (2012).
Patch ideals and peterson varieties.
Transformation groups, 17(4):1011–1036.

Kostant, B. (1996).
Flag manifold quantum cohomology, the toda lattice, and the representation with highest weight ρ.
Selecta Mathematica, 2(1):43–91.

Shimomura, N. (1980).
A theorem on the fixed point set of a unipotent transformation on the flag manifold.
Journal of the Mathematical Society of Japan, 32(1):55–64.

Spaltenstein, N. (1976).